The shape of dunes

2021 – 2022

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Research topic

After pouring the maximum quantity of sand on top of a horizontal, elevated support, what will the crests of the thereby formed *dunes* look like?

1 Theoretical basis

Since in this problem a physical phenomenon occurs, our study is based on an initial assumption that enabled us to transpose a realistic phenomenon into a mathematically described one.

Constant angle assumption The slope, denoted α, of the *dune* has the same value in any point on its surface, regardless of being closer or farther away from its top. Below you can see a vertical section through the dune. [1]

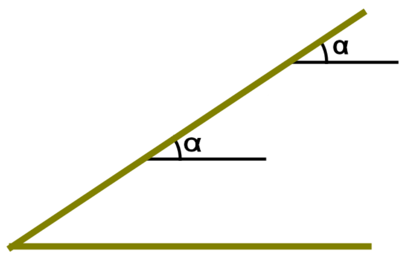

Figure 1: A vertical section through the dune

In order to study how the mathematical conditions are to be set in general, we took a simple shape, an acute angle infinitely long (see figure 2). The edges of the support are blue in the drawing, while the crest resulted after pouring the maximum quantity of sand above the support is shown in red.

2D equation If we take an arbitrary point P on the crest and its projection on the base plane, finding the mathematical equation of the crest in 2D actually means finding the equation of VQ. The projections of P on the edges of the support are M and N, respectively. Since PQ is perpendicular to the plane of the support, the projections of Q on the edges of the support will be M and N as well.

By doing this, we obtained the dihedral angles which give the slope of the *dune*: $\measuredangle PMQ$ and $\measuredangle PNQ$, which are equal, according to the *Constant angle assumption*. Because in the right triangles $\triangle PQM$ and $\triangle PQN$, PQ is a common side and also the angles at the bottom are equal, these triangles are congruent. Hence, $QM = QN$. In other words, the projection of the crest is equally away from the edges.

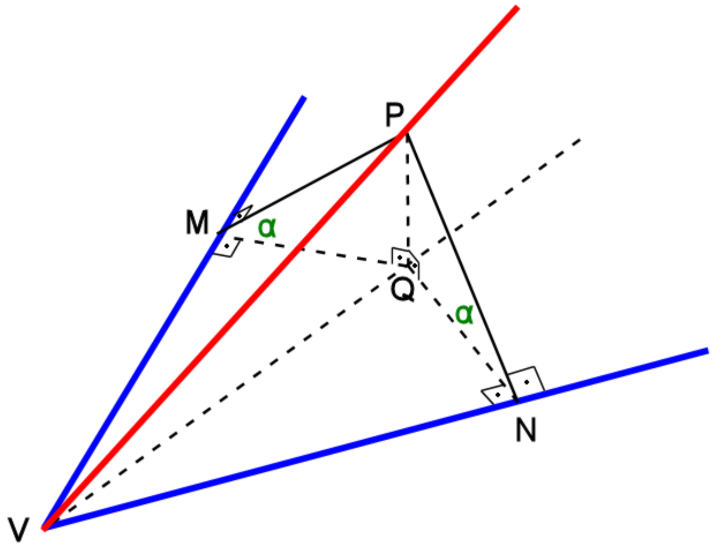

Figure 2: Acute angle support

Conclusion The mathematical equation of the crest in 2D for any given support shape can be obtained by setting the condition that it must be equally distanced from the edges of the support. This enables us to obtain the equation of crests for strange support shapes, even curved ones.

3D equation The height of point P, denoted z is determined from one of the triangles, such as $\triangle PQM$, to be $z = PQ = MQ \tan\alpha$. Therefore, in general, the height of the crest is the distance to the edge multiplied by the tangent of the slope, considered a constant value.

2 Experimental approach

In order to prove that our initial assumption offers a close enough model of the reality, we performed a series of experiments.

2.1 Triangle experiment

At first, we poured sand over the base of a triangle in order to see the crest lines (see figure 3). Even if it is approximate, the projections of the crest lines look as provided by the hypothesis lines. This result confirms that the solution of our topic is based on equidistance for the sand pile to the edges in order to be physically stable.

2.2 L-shape

Then we poured sand on a non-convex shape: The L (see figure 4). We were surprised by the rounded shape we get in the center. However, this is understandable acknowledging that in

Figure 3: The triangle experiment

this case the central part of the dune is not anymore between two lines, but between a line and a point, because in the center of the figure the sand can fall either over one of the outer sides, which are segments, or over the inner point of the corner.

Figure 4: The L-shape experiment

2.3 Parabola

In order to find out whether our theoretical basis works also in the case we encountered in the previous experiment, we performed another experiment, consisting in pouring sand on a surface bounded by a linear edge and a point (see figure 5). Beforehand, however, we drew the expected crest, which in this case is a parabola, the curve equally distanced from a line and a point. In order to see it clearly, we raised sticks prom point to point.

Conclusion As seen in the experiments, our hypothesis is an appropriate approximation of reality and we can therefore use it in out following research.

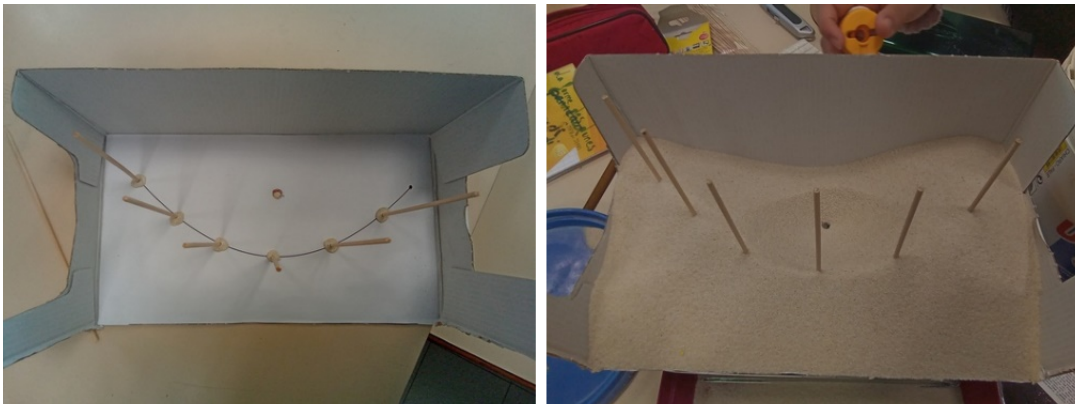

Figure 5: The parabola experiment

3 Polygonal shapes

3.1 Triangle

The support is a triangle, determined by three points. For simplicity, their respective coordinates were chosen $A(x_A, y_A)$, $B(0,0)$ and $C(x_C, 0)$. The lengths of the sides of the triangle can

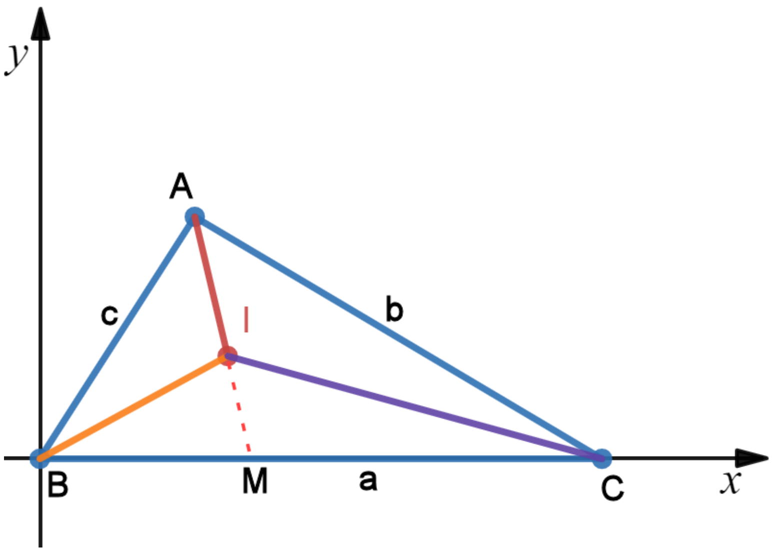

Figure 6: The triangle

be written as

$$a = x_c$$

$$b = \sqrt{(x_A - x_C)^2 + y_A^2}$$

$$c = \sqrt{x_A^2 + y_A^2}$$

The equations of the edges of the support can be written as [2]

$$BC: y = 0, \quad 0 \leqslant x \leqslant x_C$$

$$AB: xy_A - yx_A = 0 \Rightarrow AB: y = \tfrac{y_A}{x_A}x, \quad 0 \leqslant y \leqslant y_A$$

$$AC: -xy_A + y(x_A - x_C) + x_C y_A = 0 \Rightarrow AC: y = \tfrac{y_A}{x_A - x_C}x - \tfrac{x_C y_A}{x_A - x_C}, \quad 0 \leqslant y \leqslant y_A$$

The intersection point of the bisectors in a triangle, that in this case is the intersection of the three crest, was labelled I. The point lying on the intersection between AI and BC was considered M. The bisector theorem, applied in triangle $\triangle ABC$ gives

$$\frac{MC}{BM} = \frac{b}{c} \Rightarrow \frac{MC}{BM} + 1 = \frac{b}{c} + 1 \Rightarrow \frac{a}{BM} = \frac{b+c}{c} \Rightarrow BM = \frac{ac}{b+c} \tag{1}$$

Therefore, taking into account that $x_B = 0$ and $M \in Ox$, we can write

$$x_M = \frac{ac}{b+c} \tag{2}$$

Applying the bisector theorem again in triangle $\triangle ABM$ gives us

$$\frac{AI}{IM} = \frac{c}{BM} = k \tag{3}$$

k is the factor of division of AM segment and can be obtained from equations (1) and (3) as

$$k = \frac{b+c}{a} \tag{4}$$

Using the vectorial formula of dividing a segment in a given fraction gives us [3]

$$\vec{BI} = \frac{1}{k+1}\vec{BA} + \frac{k}{k+1}\vec{BM} \tag{5}$$

By projecting equation (5) onto the Ox and Oy axes, using equation (2) and taking into consideration that the coordinates of point B are $(0,0)$ and that $y_M = 0$, we obtain

$$x_I = \frac{1}{k+1}x_A + \frac{k}{k+1}\frac{ac}{b+c} = \frac{ax_A + cx_C}{a+b+c}$$

$$y_I = \frac{1}{k+1}y_A = \frac{ay_A}{a+b+c}$$

which give us the coordinates of point I as

$$I\Big(\frac{ax_A + cx_C}{a+b+c}, \frac{ay_A}{a+b+c}\Big)$$

Using the formula of a line that passes through two points and applying it for the pairs of points (A, I), (B, I) and (C, I), whose coordinates are known, we obtain

$$AI : \frac{x - x_A}{x_I - x_A} = \frac{y - y_A}{y_I - y_A} \Rightarrow AI : x(-by_A - cy_A) - y(cx_C - bx_A - cx_A) + cx_C y_A = 0$$

$$AI : \frac{x - x_B}{x_I - x_B} = \frac{y - y_B}{y_I - y_B} \Rightarrow BI : xay_A - y(ax_A + cx_C) = 0$$

$$AI : \frac{x - x_C}{x_I - x_C} = \frac{y - y_C}{y_I - y_C} \Rightarrow CI : xay_A - y(ax_A - ax_C - bx_C) - ax_C y_A = 0$$

Rearranging the equations and setting the appropriate conditions with regards to I gives us

$$AI : y = \frac{-by_A - cy_A}{cx_C - bx_A - cx_A} x + \frac{cx_C y_A}{cx_C - bx_A - cx_A}, \quad \frac{ay_A}{a+b+c} \leqslant y \leqslant y_A$$

$$BI : y = \frac{ay_A}{ax_A + cx_C} x, \quad 0 \leqslant x \leqslant \frac{ax_A + cx_C}{a+b+c}$$

$$CI : y = \frac{ay_A}{ax_A - ax_C - bx_C} x - \frac{ax_C y_A}{ax_A - ax_C - bx_C}, \quad \frac{ax_A + cx_C}{a+b+c} \leqslant x \leqslant x_C$$

which describe the equations of the crests that form over a triangular support. [4]

3.2 Square

Parameter:

1. side length - l

In this case, the support is a square determined by points with coordinates chosen for simplicity $A(0,0)$, $B(l,0)$, $C(l,l)$ and $D(0,l)$. The equations of the edges are

$$\begin{aligned} AB &: y = 0, \quad 0 \leqslant x \leqslant l \\ BC &: x = l, \quad 0 \leqslant y \leqslant l \\ CD &: y = l, \quad 0 \leqslant x \leqslant l \\ AD &: x = 0, \quad 0 \leqslant y \leqslant l \end{aligned}$$

Because the bisectors in a square coincide with the diagonals, the equations of the crests can be written as

$$AC : \frac{x - 0}{l - 0} = \frac{y - 0}{l - 0} \Rightarrow AC : xl = yl$$

which gives us

$$AC : y = x, \quad 0 \leqslant x \leqslant l \tag{6}$$

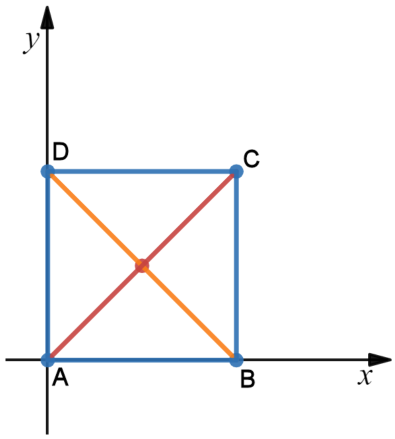

Figure 7: The square

and

$$BD : \frac{x-0}{l-0} = \frac{y-l}{0-l} \Rightarrow BD : -xl = l(y-l)$$

which gives us

$$BD : y = l - x, \quad 0 \leqslant x \leqslant l \tag{7}$$

Equations (6) and (7) describe the crests that form in this case.

3.3 Rectangle

Parameters:

1. length - L
2. width - l

In this case, the support is a rectangle determined by points with coordinates chosen for simplicity $A(0,0)$, $B(L,0)$, $C(L,l)$ and $D(0,l)$. The two points where groups of three crests intersect were labelled P and Q. The equations of the edges are

$$\begin{aligned} AB &: y = 0, \quad 0 \leqslant x \leqslant L \\ BC &: x = L, \quad 0 \leqslant y \leqslant l \\ CD &: y = l, \quad 0 \leqslant x \leqslant L \\ AD &: x = 0, \quad 0 \leqslant y \leqslant l \end{aligned}$$

Since the crest is the line that is equally distanced from two edges, as demonstrated in the theoretical basis, the PQ segment of the crest has the equation

$$PQ : y = \frac{l}{2} \tag{8}$$

Since P also belongs to the crests DP and AP, it means that the distance from P to AB, which is $\frac{l}{2}$, as given by the previous equation, is equal to the distance from P to the segment AD,

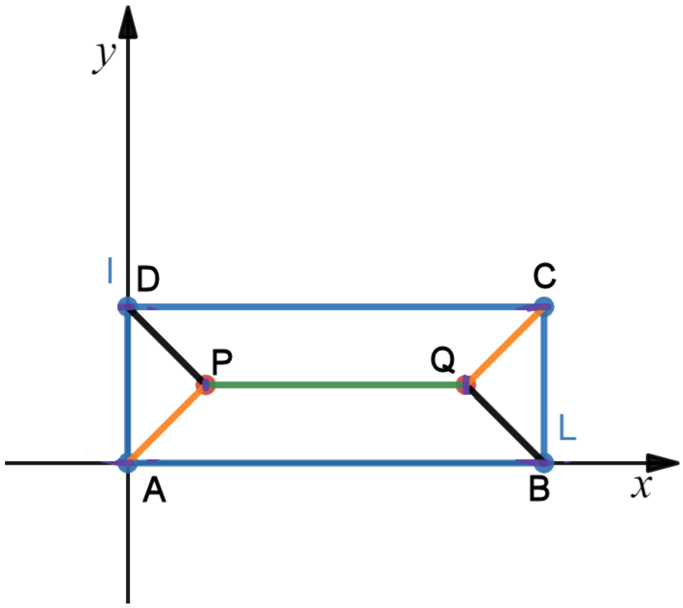

Figure 8: The rectangle

meaning that P is of coordinates $P(\frac{l}{2},\frac{l}{2})$. Through an analogue reasoning, the equations of Q are obtained as $Q(L-\frac{l}{2},\frac{l}{2})$. Now that all points have determined coordinates, the other four crests can be determined using the formula of a line that passes through two points in the plane

$$AP: \frac{x-0}{l/2-0} = \frac{y-0}{l/2-0} \Rightarrow AP: y = x, \quad 0 \leqslant x \leqslant \frac{l}{2}$$

$$DP: \frac{x-0}{l/2-0} = \frac{y-l}{l/2-l} \Rightarrow DP: y = -x+l, \quad 0 \leqslant x \leqslant \frac{l}{2}$$

$$CQ: \frac{x-L}{L-l/2-L} = \frac{y-l}{l/2-l} \Rightarrow AP: y = x-L+l, \quad L-\frac{l}{2} \leqslant x \leqslant L$$

$$BQ: \frac{x-l}{L-l/2-L} = \frac{y-0}{l/2-0} \Rightarrow AP: y = -x+L, \quad L-\frac{l}{2} \leqslant x \leqslant L$$

and also equation (8) can be restricted to the segment PQ and written as

$$PQ: y = \frac{l}{2}, \quad \frac{l}{2} \leqslant x \leqslant L-\frac{l}{2}$$

3.4 L-shape

The L-shape is a polygon $ABCDEF$. Its dimensions were taken L_1, L_2, l_1 and l_2, as in figure 9. The coordinates of these points were taken as $A(0,0)$, $B(L_1,0)$, $C(L_1,l_1)$, $D(l_2,l_1)$, $E(l_2,L_2)$ and $F(0,L_2)$. In an L-shaped support intervene many crests, many of which have the same equations as in the rectangle shape and have therefore not been analysed anymore. In the corner of L, the shape of the crest has been carefully studied. As observed in the experiments, two distinct curves form here. Their point of intersection was marked K and represents the point that is equally distanced from the three sides where sand can fall: AB edge, AF edge and point D. An arbitrary point of coordinates (x, y) is on the curve between S and K if the

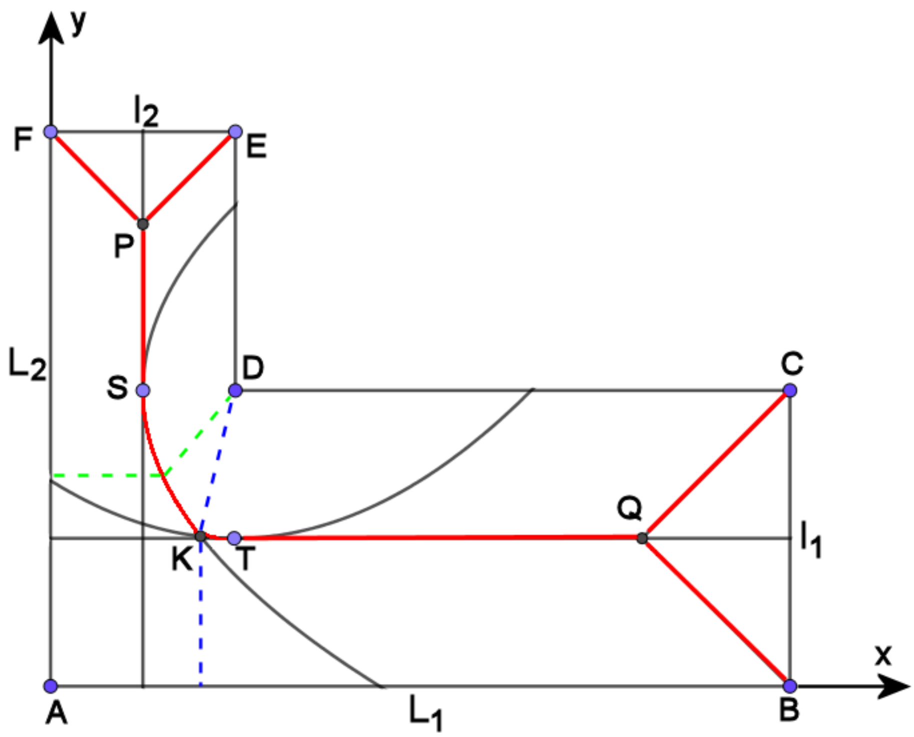

Figure 9: The L-shape

distance from it to the AF edge, which is x, equals the distance from it to point D, which, assuming that D is of coordinates $D(l_2, l_1)$ can be written as

$$\sqrt{(l_1 - y)^2 + (l_2 - x)^2} \tag{9}$$

Equalising the two distances and doing the calculations gives us the equation of the curve

$$x = \frac{1}{2l_2}y^2 - \frac{l_1}{l_2}y + \frac{l_1^2 + l_2^2}{2l_2} \tag{10}$$

which is the equation of a parabola, however rotated by 90°. [5]

An arbitrary point of coordinates x and y is on the curve between K and T if the distance from it to the AB edge, which is y, equals the distance from it to point D, which can be written as in equation (9). Equalising the two distances and doing the calculations give us the equation of the curve

$$y = \frac{1}{2l_1}x^2 - \frac{l_2}{l_1}x + \frac{l_1^2 + l_2^2}{2l_1} \tag{11}$$

that is also a parabola.

4 Circular shapes

The last category of shapes that we have studied are curved-shaped supports. Covering all of them from a simple half disc to a circle from which two inner circles have been cut out, they are presented in the order of complexity.

4.1 Half disc

Parameters:

1. radius of the disc - r

For simplicity, the sysem of axes has been chosen so that its origin coincides with the center of the disc and the Ox axis is overlapped with the straight edge. An arbitrary point $P(x, y)$ is

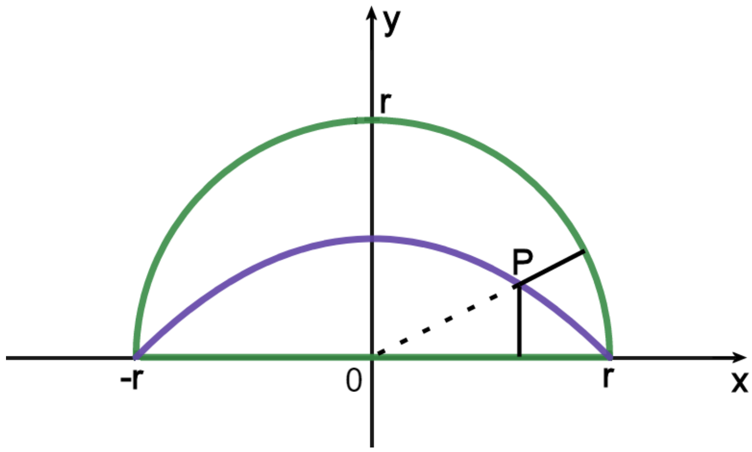

Figure 10

on the crest if the distance from P to the straight edge is equal to the distance to the curved edge. The first of this distances can be written as

$$y$$

while drawing the radius of the circle that passes through P and taking into account that the radius of a circle is perpendicular to its outline, the second distance can be written as

$$r - \sqrt{x^2 + y^2}$$

Equalising the two quantities gives us

$$y = r - \sqrt{x^2 + y^2} \Leftrightarrow \sqrt{x^2 + y^2} = r - y \Rightarrow x^2 + y^2 = r^2 - 2ry + y^2 \Leftrightarrow 2ry = -x^2 + r^2$$

Finally, restricting the domain of our equation to $[-r, r]$, so that we have crest only inside the shape, the equation of the crest is as follows

$$y = -\frac{1}{2r}x^2 + \frac{1}{2}r, \quad -r \leqslant x \leqslant r \tag{12}$$

which is the equation of a parabola.

4.1.1 3D expansion

As demonstrated in the theoretical basis, the height of the crest is the distance to the edge multiplied by the tangent of the slope. It is correct to use any of the distances to the edge, but for simplicity, we can take the distance to the straight edge, which is y. Therefore, the z equation of point P can be written as

$$z = y \tan\alpha \tag{13}$$

Combining equations (12) and (13) gives us a system of equations that describes the crest in 3 dimensions

$$\begin{cases} y = -\frac{1}{2r}x^2 + \frac{1}{2}r \\ z = y \tan\alpha \end{cases}, \quad -r \leqslant x \leqslant r$$

4.2 Partially cut disc

This shape is a disc in which a decentered line was drawn and the part on one side of the line has been removed.

Parameters:

1. radius of the disc - r
2. distance from the center of the circle to the cut - d, with $d < r$

Similarly, a system of equations was chosen so that the Ox axis is overlapped with the straight edge and the center of the disc being situated on the Oy axis. An arbitrary point $P(x, y)$ is on the crest if the distance from P to the straight edge is equal to the distance to the curved edge. The first of this distances can be written as

$$y$$

while drawing the radius of the circle and taking into account that the radius of a circle is perpendicular to its outline, the second distance can be written as

$$r - \sqrt{x^2 + (y-d)^2}$$

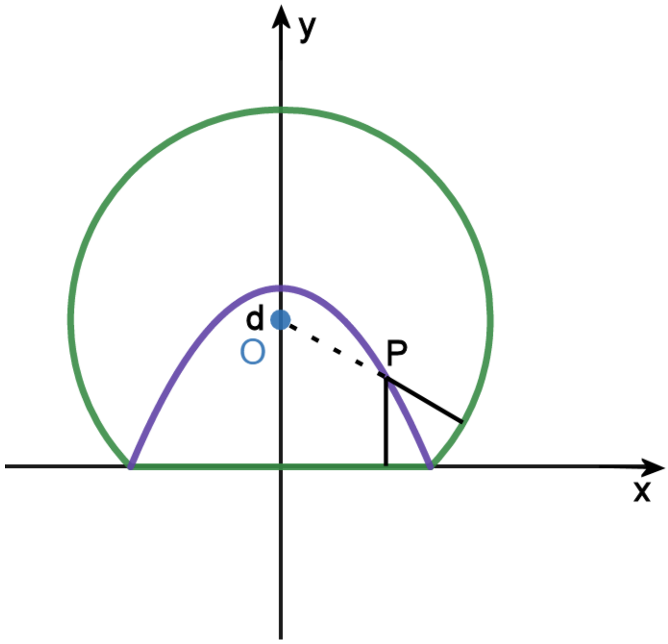

Figure 11: Partially cut disc

Equalising the two quantities gives us

$$y = r - \sqrt{x^2 + (y-d)^2} \Leftrightarrow \sqrt{x^2 + (y-d)^2} = r - y \Rightarrow$$

$$\Rightarrow x^2 + y^2 - 2dy + d^2 = r^2 - 2ry + y^2 \Leftrightarrow 2y(r-d) = -x^2 + r^2 - d^2$$

Finally, the equation of the crest is as follows

$$y = -\frac{1}{2(r-d)}x^2 + \frac{r+d}{2} \tag{14}$$

which is also a parabola [6]. From the plot we observe that the restriction we must set in order to obtain the proper equation is $y \geqslant 0$, which substituted in equation (14) gives

$$-\frac{1}{2(r-d)}x^2 + \frac{r-d}{2} \geqslant 0 \Rightarrow x^2 < r^2 - d^2 \Rightarrow x \in [-\sqrt{r^2-d^2}, \sqrt{r^2-d^2}]$$

4.2.1 3D expansion

As demonstrated in the theoretical basis, the height z of the point P equals the distance to one edge (let's take the distance to the straight edge, for simplicity) times the slope

$$z = y \tan \alpha \tag{15}$$

Combining equations (14) and (15) gives us a system of equations that describes the crest in 3 dimensions

$$\begin{cases} y = -\dfrac{1}{2(r-d)}x^2 + \dfrac{r+d}{2} \\ z = y \tan\alpha \end{cases}, \quad -\sqrt{r^2-d^2} \leqslant x \leqslant \sqrt{r^2-d^2}$$

4.3 Crescent

The crescent is a shape that is determined by two circles.

Parameters:

1. radiuses of the circles - r_1, r_2
2. distance between the centres of the two circles - d, with $d < r_1 + r_2$

The system of axes has been chosen so that its origin coincides with the center of the circle of radius r_1, and the Oy axis passes through the center of the circle of radius r_2. An arbitrary

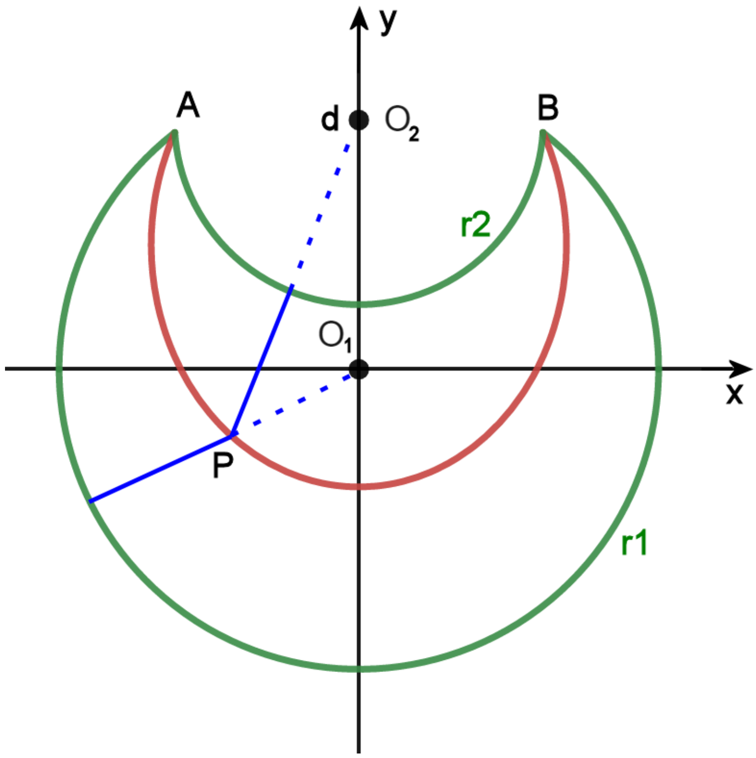

Figure 12: The crescent

point $P(x, y)$ is on the crest if the distances from P to the two circles are equal. Assuming that in any circle the radius is perpendicular to the outline [7] , by drawing the radiuses of the two circles that pass through P, we obtain the distance from P to the circle of radius r_2

$$\sqrt{x^2 + (y-d)^2} - r_2$$

and the distance from P to the circle of radius r_1

$$r_1 - \sqrt{x^2 + y^2}$$

Setting the equality condition between the two, we obtain the equation of the crest for the crescent, in implicit form

$$\sqrt{x^2 + (y-d)^2} - r_2 = r_1 - \sqrt{x^2 + y^2} \tag{16}$$

The restriction that applies here so that the crest does not exceed the limits of the support, is $y \leqslant y_{AB}$.

4.3.1 3D expansion

As demonstrated in the theoretical basis, the height z of the point P can be written as the horizontal distance to the edge times the slope of the dune

$$z = (r_1 - \sqrt{x^2 + y^2}) \tan \alpha \tag{17}$$

In order to obtain the correct restriction for this crest, so that it does not exceed the terminus points of the crescent (denoted A and B in figure 12), we have to find the y coordinate of points A and B, which, due to symmetry, are equal. These points fulfill the property that they lay on both the two circles that make up the support, therefore we can write

$$\begin{cases} r_1^2 = x^2 + y^2 \\ r_2^2 = x^2 + (y-d)^2 \end{cases}$$

Solving this system of equations gives

$$y = \frac{d^2 + r_1^2 - r_2^2}{2d} \tag{18}$$

Combining equations (16) and (17) and restricting to $y \leqslant y_{AB}$, according to equation (18), gives us a system of equations that describes the crest of the dune that forms over a crescent-shaped support

$$\begin{cases} \sqrt{x^2 + (y-d)^2} - r_2 = r_1 - \sqrt{x^2 + y^2} \\ z = (r_1 - \sqrt{x^2 + y^2}) \tan \alpha \end{cases} , \quad y \leqslant \frac{d^2 + r_1^2 - r_2^2}{2d}$$

4.3.2 Drilled circle

Another interesting shape to study is a circle from which another smaller circle inside has been cut out. However, this shape is an alteration of the crescent, obtained by simply reducing the r_2 radius so that $r_2 < r_1$ and ensuring that the entire circle of radius r_2 fits within the

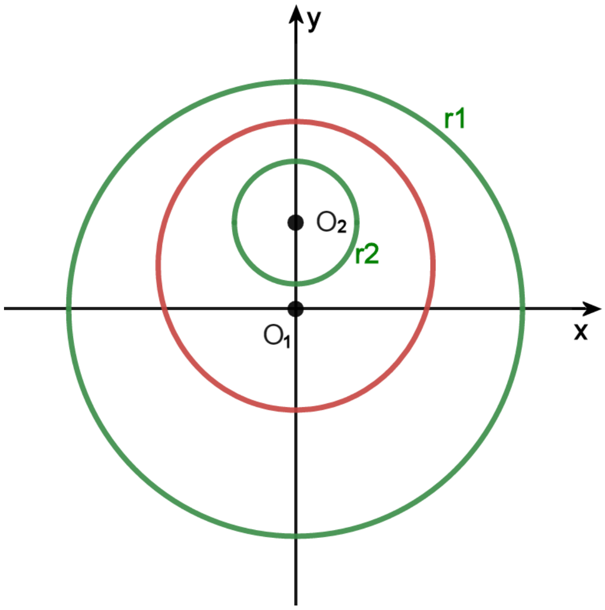

Figure 13: Drilled circle

other one. Since only numerical values have been modified, the mathematical equations that describe the crest remain the same as for the crescent, even in 3 dimensions, with the mention that the equations do not have to be restricted anymore, because in this case, the crest is continuous, surrounding the smaller circle completely.

$$\begin{cases} \sqrt{x^2+(y-d)^2}-r_2 = r_1-\sqrt{x^2+y^2} \\ z = (r_1-\sqrt{x^2+y^2})\tan\alpha \end{cases}$$

4.4 Double-drilled circle

The last and most complex shape that is covered in this article is a circle from which two smaller circles inside have been cut out, fulfilling the property that their respective centers are collinear.

Parameters:

1. radius of the big circle - R
2. radiuses of the cuts - r_1 and r_2, with $r_1 + r_2 < R$
3. distance between the centres of the two small circles to the center of the big circle - d_1 and d_2, with $d_1 < R - r_1$ and $d_2 < R - r_2$

In this case, the system of equations has been chosen so that its origin coincides with the center of the big circle and the Oy axis passes through the centers of all circles. As observed in the practical experiments, three distinct crests form, that all intersect in two points, labelled $A(x_A, y_A)$ and $B(x_B, y_B)$. The crests were labelled a, b and c, as in figure 14. Due to symmetry, the two points fulfill the property that $y_A = y_B$.

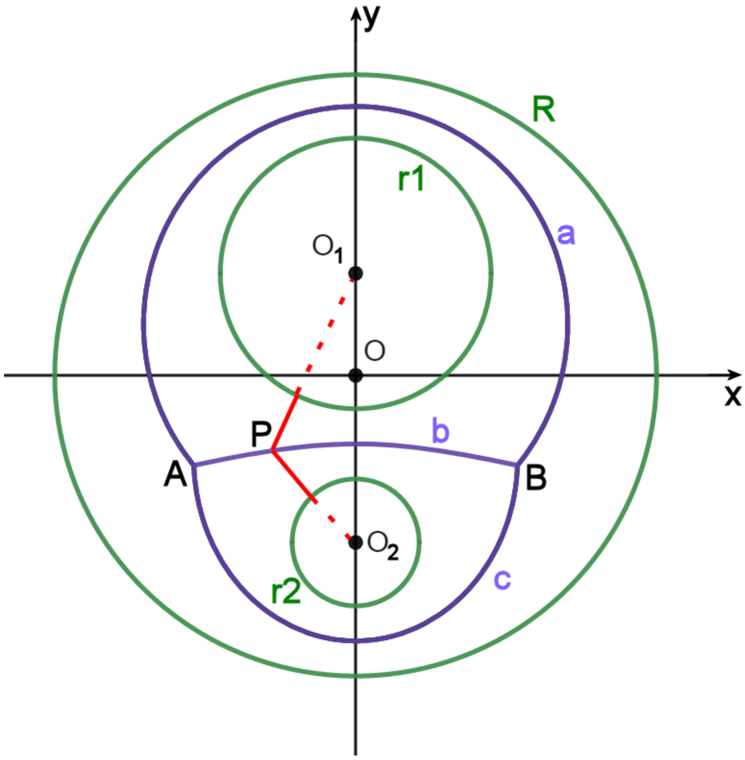

Figure 14: Double drilled circle

The upper and lower crests were studied in the crescent shape, because they stand between two circles. However, they have to be restricted according to the points A and B. Hence, taking into consideration that now the radiuses have been labelled differently, crests (a) and (c) are described by the following systems of equations

$$(a)\begin{cases}\sqrt{x^2+(y-d_1)^2}-r_1 = R-\sqrt{x^2+y^2} \\ z=(R-\sqrt{x^2+y^2})\tan\alpha\end{cases}, y \geqslant y_A$$

$$(c)\begin{cases}\sqrt{x^2+(y-d_2)^2}-r_2 = R-\sqrt{x^2+y^2} \\ z=(R-\sqrt{x^2+y^2})\tan\alpha\end{cases}, y \leqslant y_A$$

As for the crest (b), its equation is similar to the others, with the exception that in this case, the crest is outside both circles, unlike the previously studied crests, that were inside one circle and outside the other. The distance from P to the edge of the circle of radius r_1 can be written as

$$\sqrt{x^2+(y-d_1)^2}-r_1$$

and the distance from P to the edge of the circle of radius r_2 can be written as

$$\sqrt{x^2+(y-d_2)^2}-r_2$$

Equalizing these two equations gives us

$$\sqrt{x^2+(y-d_1)^2}-r_1=\sqrt{x^2+(y-d_2)^2}-r_2 \tag{19}$$

Applying the same procedure as in the previously studied shapes, we obtain the height z of the crest as

$$z=(\sqrt{x^2+(y-d_1)^2}-r_1)\tan\alpha \tag{20}$$

Combining equations (19) and (20) and setting the proper restrictions gives us

$$(b)\begin{cases}\sqrt{x^2+(y-d_1)^2}-r_1=\sqrt{x^2+(y-d_2)^2}-r_2\\ \\ z=(\sqrt{x^2+(y-d_1)^2}-r_1)\tan\alpha\end{cases},x\in[x_A,x_B]$$

Conclusion

In conclusion, our study focused only on certain categories of shapes from the multitude that exist. We chose to study shapes bounded by lines and/or circles because in these cases the equidistance conditions are straightforward and do not require complex calculus knowledge. However, this study could be extended to other types of edges, such as: sinusoidals, exponentials, elliptics, parabolas or hyperbolas in order to provide a more complete solution to the problem of dunes.

References

[1] Desmos, *Graphing calculator*, https://www.desmos.com/calculator, accessed on 1st June 2022

EDITION NOTES

[1] It should also be made explicit that, at any point of the surface, the maximum slope direction is assumed to be toward the nearest point of the edge of the support. This assumption has a crucial role in the work. A consequence of it is that the maximum slope direction is perpendicular to the tangent to the edge, if such a tangent exists.

[2] In the work, given two points X and Y in the plane, the symbol XY is used in three different ways. It denotes either the segment, or its length, or the straight line passing through the two points. This causes no confusion because the context always clarifies the meaning of the symbol. However, this slight abuse of notation should be pointed out explicitly.

[3] Some explanation about this equality would be advisable. It should have been specified that it derives from the vector equalities $\vec{IB}+\vec{BA}=\vec{IA}=k\vec{MI}=k(\vec{MB}+\vec{BI})$.

[4] The equation of the straight lines AI, BI, and CI can also be determined in another way. Perhaps it is simpler than that considered in the work. We can use the formula for the distance between a point $P_0(x_0,y_0)$ and the straight line of equation $\alpha x+\beta y+\gamma=0$: $\dfrac{|\alpha x_0+\beta y_0+\gamma|}{\sqrt{\alpha^2+\beta^2}}$.
If we consider this formula with the equations of the lines AB and BC, an arbitrary $P(x,y)$ verifies the equation $\dfrac{|y_A x-x_A y|}{\sqrt{y_A^2+x_A^2}}=\dfrac{|y|}{1}$ if and only if it is equally distanced from AB and BC, which means that P belongs to one of the two bisectors of the angle $A\hat{B}C$.
Since $\sqrt{y_A^2+x_A^2}=c$, these two bisectors have equation $y_A x-x_A y=cy$ and $y_A x-x_A y=-cy$, that is $y=\dfrac{y_A}{c+x_A}x$ and $y=\dfrac{y_A}{-c+x_A}x$. The equation of BI is the first one because it has a positive slope. It coincides with the one given in the work because $x_c=a$.
The equation of CI can be found in the same way, using the formula of the distance with the equations of AB and BC : $\dfrac{|y_A x+(x_C-x_A)y-x_C y_A|}{\sqrt{y_A^2+(x_C-x_A)^2}}=\dfrac{|y|}{1}$, that is $y_A x+(x_C-x_A)y-x_C y_A=\pm by$ because the expression in the square root is equal to b^2.
Then, taking into account that the line CI has a negative slope, we obtain that its equation is $y=\dfrac{y_A}{x_a-x_c-b}x-\dfrac{x_C y_A}{x_a-x_c-b}$. Observe that also in this case the expression given in the work can be simplified using the equality $x_c=a$.
The point I can now be determined by intersecting BI and CI, and, for the line AI, we can use the formula for the line passing through two given points.

[5] The parabola can be defined as the locus of the points with equal distance from e point F (*focus*) and a straight line d (*directrix*). So, we can conclude that SK and KT are arcs of a parabola without expressing their equation. According to this definition, we can have parabolas with arbitrary focus and directrix, not only with directrix parallel to the x axis. Then, the specification "however rotated by 90°" is not strictly speaking correct.

[6] It should be observed here that, for $d=0$, the equation (14) becomes (12). In fact, the half cut disc is just a partially cut disc with $d=0$.

[7] This is not an assumption, it is a theorem. Presumably, the authors wanted to say "Since in any circle ...."