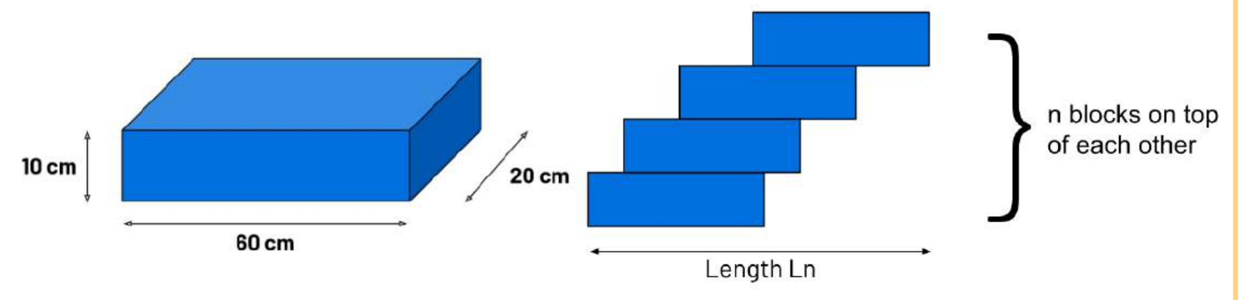
The Largest Building

by Gavrilă Ioana, Bud Sofia and Lazăr Alexandru-Florin
Colegiul Național "Emil Racoviță", Cluj-Napoca, Romania

The Research Topic



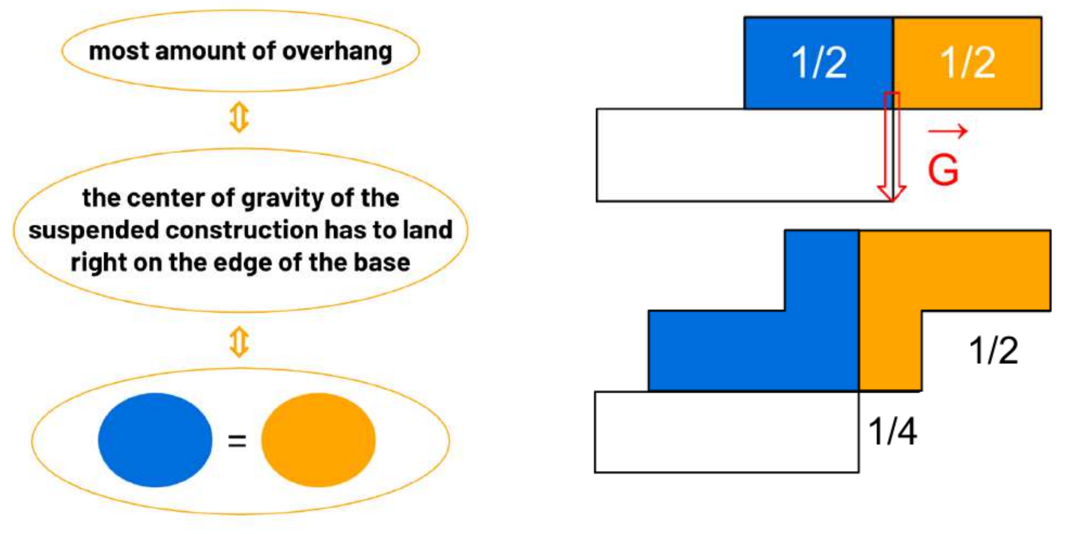
Can we place many of these rectangular blocks on top of each other without them collapsing, such that the length Ln of this construction is **10 meters** long horizontally? If yes, how many pieces do we need?

Is it possible to achieve a length Ln of **100 meters** without the blocks collapsing? What is the minimal number of pieces n necessary for such a construction?

What would the vertical height of such constructions be?

What we need to find is the relationship between a number of bricks and the maximum amount of overhang achievable with said number.

Initial Observations



Using Mathematical Induction

The maximum overhang can be interpreted as a sum:

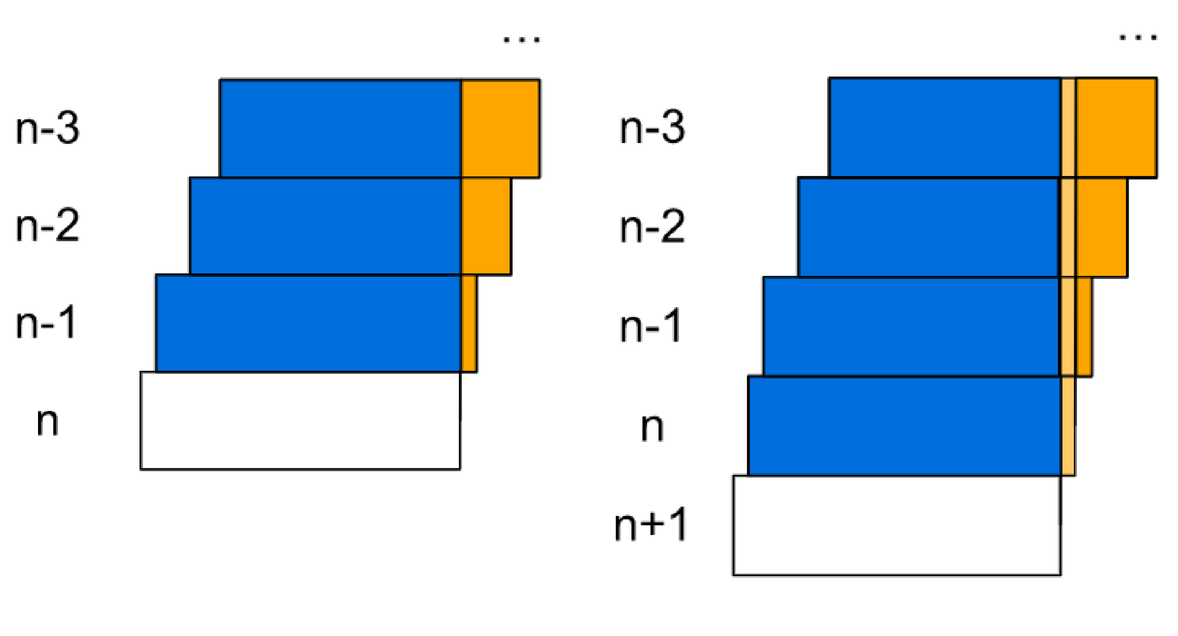
$$s(1) = \frac{1}{2}, s(2) = \frac{1}{2} + \frac{1}{4}, \ldots$$

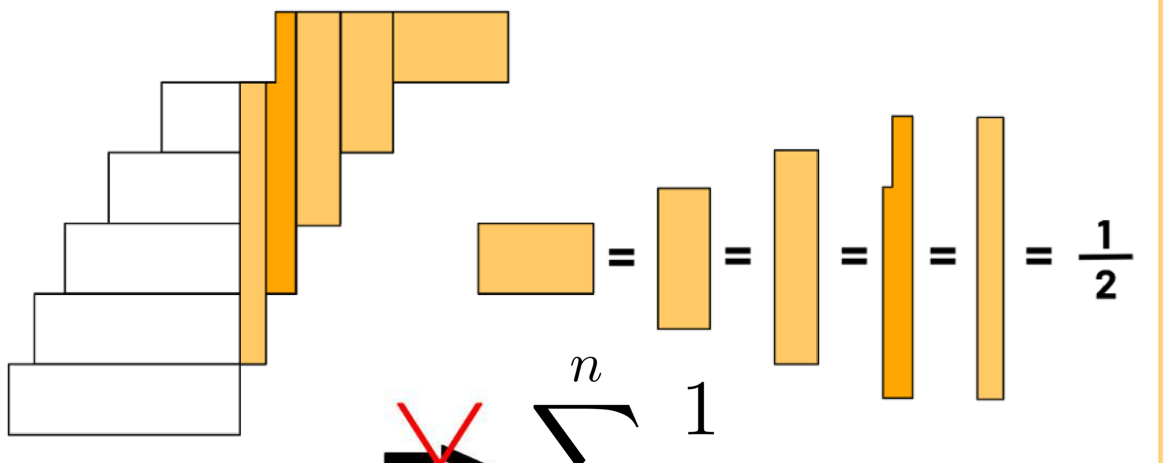
We want to prove the statement:

$$p(n): \text{“}s(n) - s(n-1) = \frac{1}{2 \cdot n}\text{”}, \forall\, n \in \mathbb{N}, n > 1$$

Base step $s(2) - s(1) = \frac{1}{2} + \frac{1}{4} - \frac{1}{2} = \frac{1}{4}$ ✓

Inductive step p(n-1) → p(n)





$$\not\Rightarrow \sum_{k=1}^{n} \frac{1}{2k}$$

In fact, our real sum will look something like this:

$$s = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{\mathbf{5}}{\mathbf{36}} + \frac{1}{8} + \frac{1}{10} + \frac{\mathbf{161}}{\mathbf{1800}} + \frac{1}{12} + \ldots$$

=> ~~$p(n): \text{“}s(n) - s(n-1) = \frac{1}{2 \cdot n}\text{”}, \forall\, n \in \mathbb{N}, n > 1$~~

However, we will proceed with using

$$\sum_{k=1}^{n} \frac{1}{2k} = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{10} + \ldots$$

for simplicity reasons, with the awareness that it is not fully accurate.

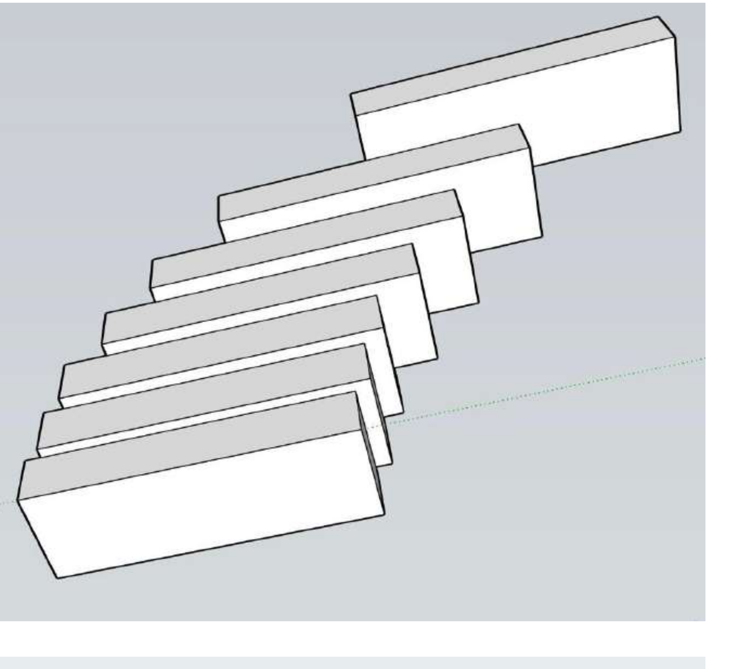
Is any amount of overhang achievable?

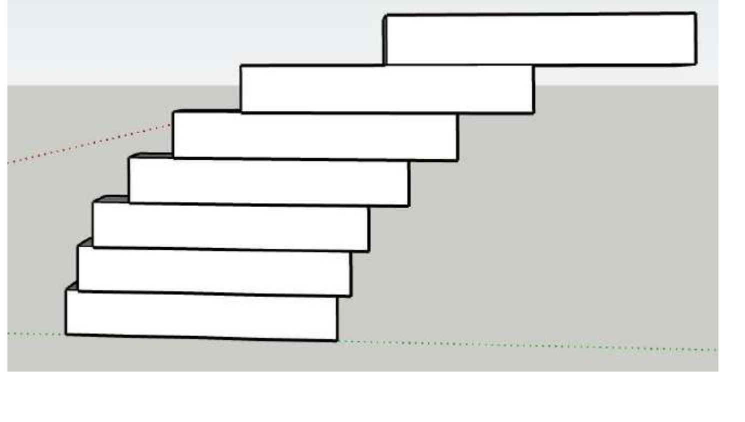
Yes, because the harmonic series, which is equal to double the value of our sum, tends to infinity.

but only in theory

Diagonal Placement

Since we are interested in obtaining as much length as possible, a diagonal placement would be more favorable.





Thus, our "length unit" equals

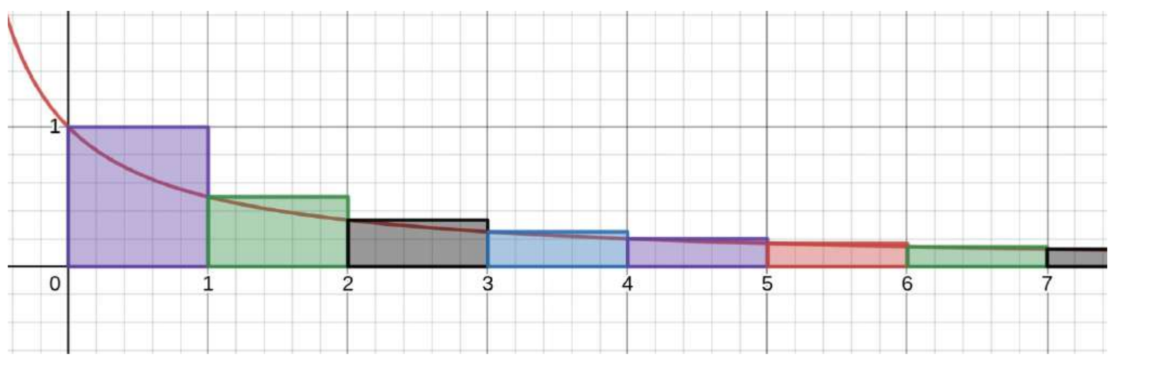
$$\sqrt{(length\ of\ block)^2 + (width\ of\ block)^2}$$

Euler demonstrated that the harmonic series can be approximated with the natural logarithm in 1735.

$$\lim_{n\to\infty} \frac{h(n)}{\ln n} = 1$$

← this statement can easily be proved with the Cesàro-Stolz theorem

This does make sense visually.



$$\lim_{n\to\infty} h(n) - \ln n = \gamma = 0.577215\ldots$$

γ is the Euler-Mascheroni constant.

Final Formula

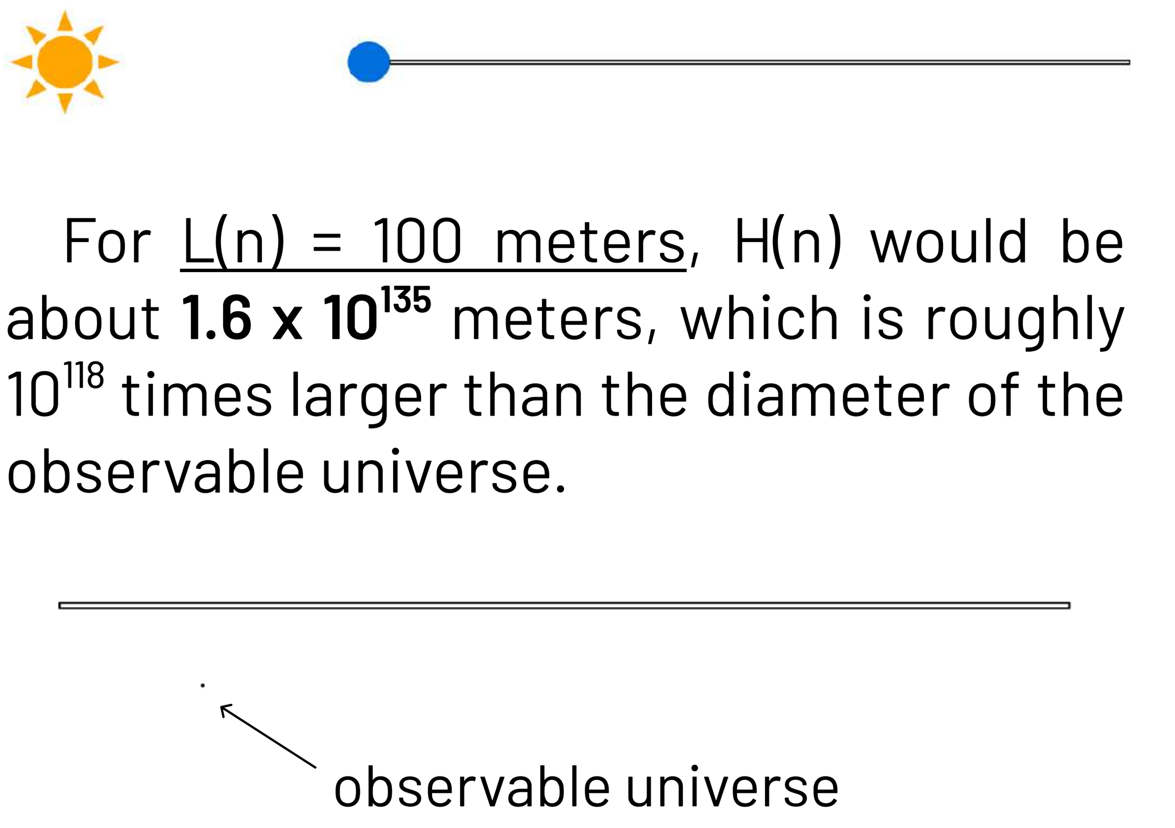
$$H(n) = h \cdot \underbrace{e^{\left[2\cdot\left(\frac{L(n)}{\sqrt{l^2+w^2}} - 1\right) - \gamma\right]}}_{n}$$

l - length of one block
w - width of one block
h - height of one block
e - Euler's number = 2,71828...

n - number of blocks
H(n) - height of building of n blocks
L(n) - length of building of n blocks
γ - the Euler-Mascheroni constant = 0,57721...

- this formula is especially accurate for greater values of n

For L(n) = 10 meters, the construction would require **4 114 592 580 543** blocks and would be 411459258054,3 meters high, which is about 2.7 times the distance form Earth to the Sun.



For L(n) = 100 meters, H(n) would be about **1.6 x 10^{135}** meters, which is roughly 10^{118} times larger than the diameter of the observable universe.

Considerations

- Obviously, **basic physical realities** (like wind, the lack of a strong gravitational force in space, the size of the building etc.) **inhibit us from actually building such massive buildings**.
- Because we are using the imprecise sum, this means that any overhang could theoretically be achieved faster, but not by a significant margin.

Programmes used in the making of this poster: Figma, Google Docs Drawing Tool, Overleaf, Desmos and SketchUp