# Inflated Sets

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## Understanding the problem

Let us prove that the diameter of any convex polygon is a segment that unites two vertices of the polygon. We can intuitively prove the fact that the diameter is delimited by two points on the sides of the polygon.

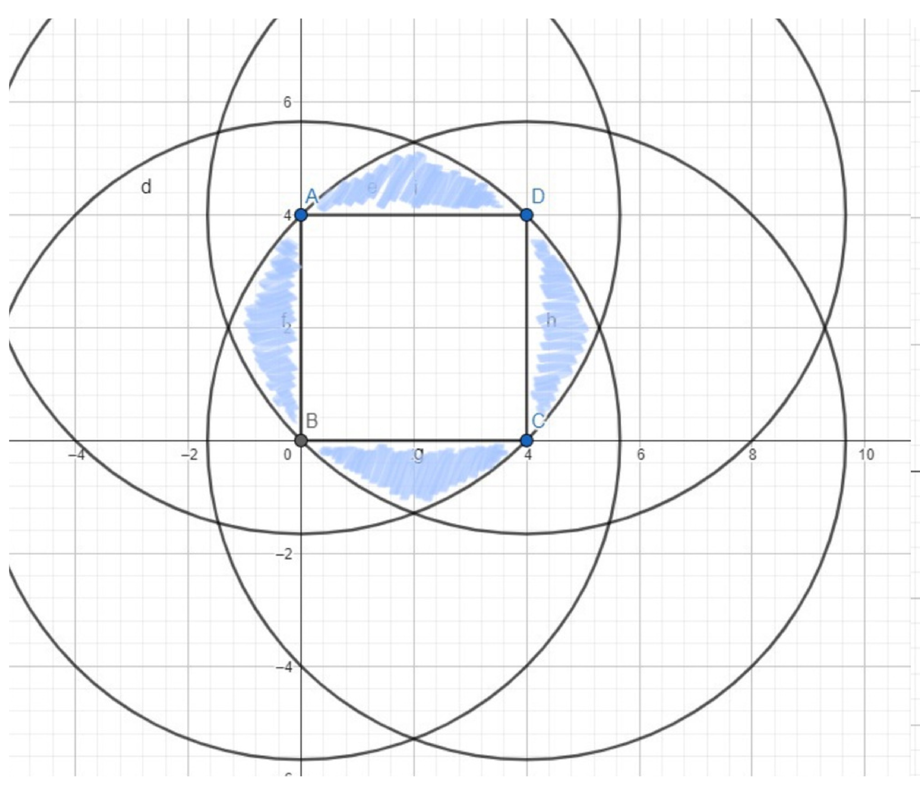
Now let there be a point A on the outline of the figure and an edge [MN]. If we arbitrarily choose a point P ∈ [MN] so that (AP ⊥ MN), one of the angles ∠APM and ∠APN will be obtuse, and the other will be acute (as they together form 180°). Should we assume that ∠APN is the obtuse angle, it will be the widest angle in △APN. Opposite the widest angle of the triangle lays the largest side, therefore AN > AP. Similarly, should ∠AMP be the obtuse angle, we obtain AM > AP, and should we choose the point P ∈ [MN] so that AP ⊥ MN, [AM] and [AN] would be the hypotenuses of the right-angled triangles △ APM and △ APN, thus AM > AP and AN > AP. In conclusion, there will always be a segment determined by two vertices longer than any segment determined by points on the polygon (excluding the vertices). Now that we have established this, we can star inflating the polygons in two different approaches:

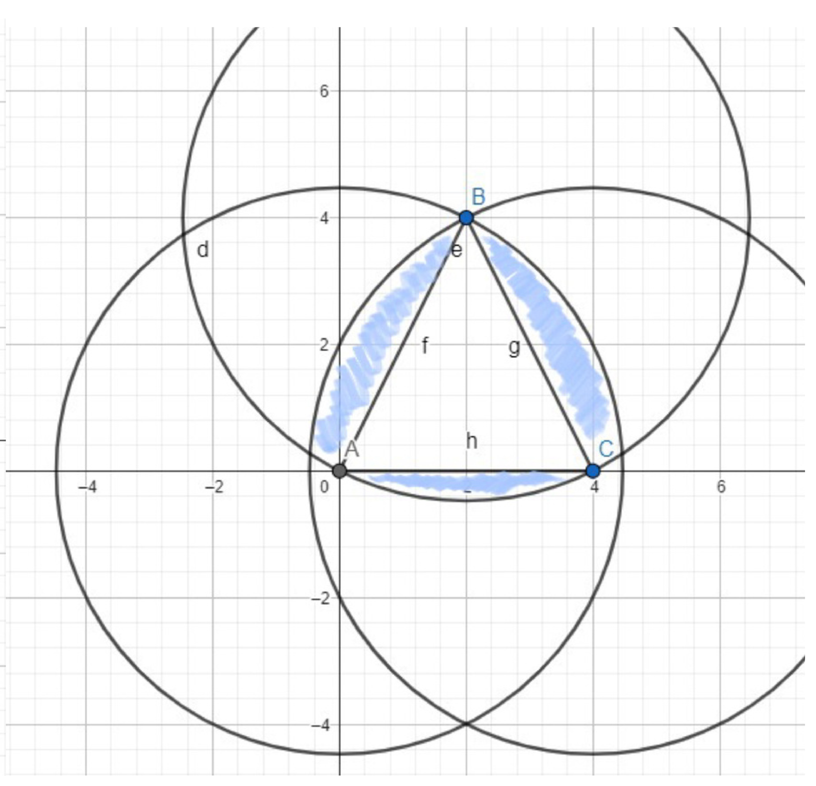
## Approach #1

*This approach is based on steps and successive choices of point in more restricted areas.*

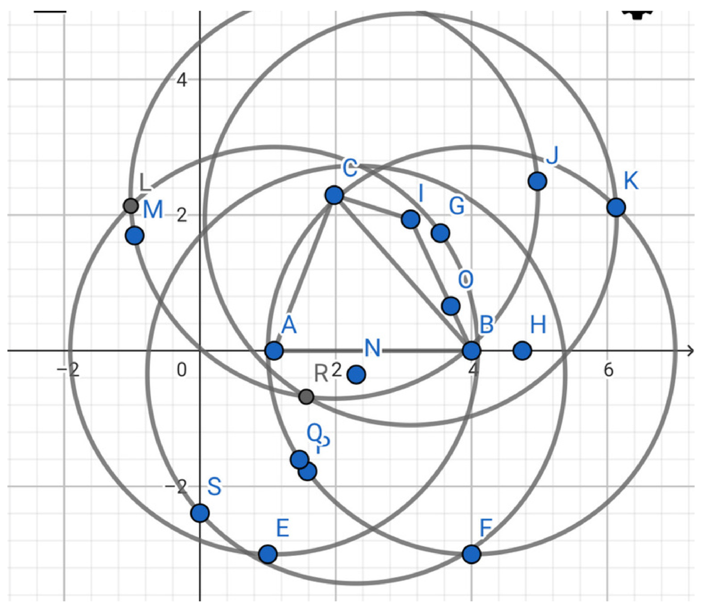
We started the process of inflating the figures by choosing different convex figures (triangles, squares etc.). We drew circles in the points of the given figures with the same diameter as the figure.

In the space created by the intersection of the circles, we chose points as far away from the sides of the figure as possible. Then again, we drew circles with the same diameter as the initial ones, with the centre being in the new points.





In this way, we ended up with some very small spaces formed by the intersection of the circles. Thus, no matter how many more points we add, there will be no change, since the distance between the points will be minimal. Now, by connecting the new points we will obtain the inflated figure.



## Introduction

We take a convex plane figure. The diameter of the figure is the greatest distance between two points of this figure.To this figure (A) we can "add" a point M outside the figure by considering the convex envelope formed by A and M. It is like placing a rubber band around A and M.

A figure is said to be "inflated" when the addition of any point in the plane (in the above sense) increases its diameter.

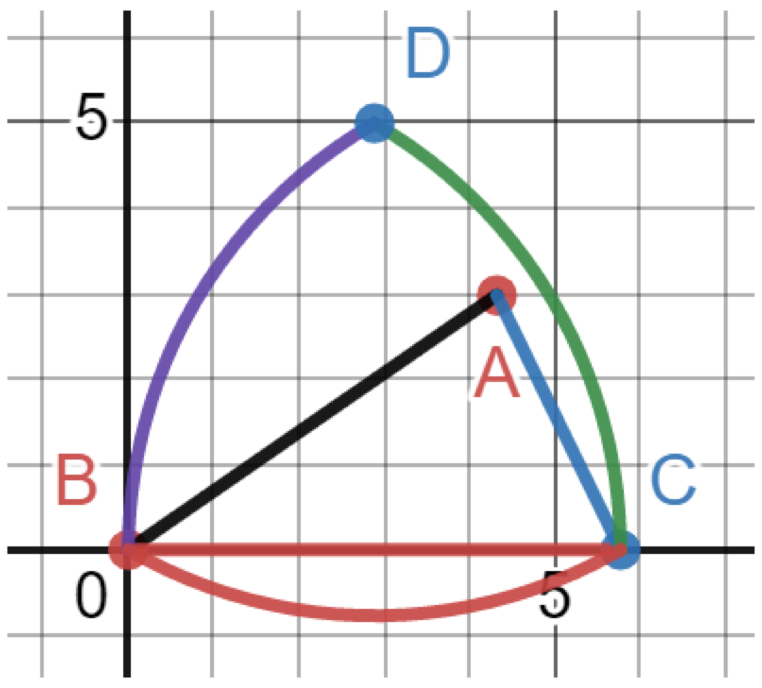
Try inflating a square and other shapes. What can be said about inflated figures of the same diameter?

## Approach #2

*This approach essentially proposes that we modify the figure, according to d, the maximum distance from a point of the figure to another point of the figure, so that each and every point of the figure has a corresponding point on the outer surface of the figure, situated at the distance d of it.* Thus, adding any other exterior points will increase the distance d, the figure therefore needing further inflation.

<u>Inflating a triangle</u>

Let us take the scalene triangle with coordinates A(a1,a2), B(0, 0) and C(c1, 0). The equations of the sides of the triangle are:




$$y=\frac{xa_2}{a_1}\{0\le y\le a_2\}$$

8 $$y=0\{0\le x\le c_1\}$$

9 $$y=\frac{(xa_2-c_1a_2)}{a_1-c_1}\{0\le y\le a_2\}$$

Step 1 – determine the maximum distance between 2 points of the figure - essentially its longest side - d. For a triangle, we shall have 3 distinct cases - the diameter can only be equal to its side of maximum length: d = BC, d = AC or d = AB.

$a=c_1$ $b=\sqrt{(a_1-c_1)^2+a_2^2}$ $c=\sqrt{a_1^2+a_2^2}$ $d\ =\ \max(a,b,c)$

Here, we shall study the case d = BC (the others are much more tedious):

Step 2 – trace 2 circles from the apexes of that side (here B and C) of length d - centers in those respective apexes and the radius d

$x^2+y^2=d^2\ \{d=a\}\left\{\frac{d}{2}\le x\right\}\{y>0\}$ $(x-d)^2+y^2=d^2\{d=a\}\left\{x\le\frac{d}{2}\right\}\{y>0\}$

Step 3 – take point D - intersection of those 2 circles - from the 2 intersection points of those circles, take the one in the same semiplane fenced by BC as A

$\left(\frac{d}{2},\frac{d\sqrt{3}}{2}\right)\{d=a\}$ Label: D

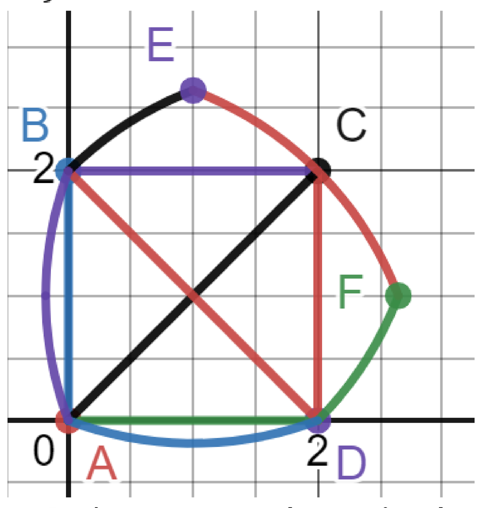
Step 4 – trace the circle with the centre in D and the radius d - arch from B to C

$$\left(x-\frac{d}{2}\right)^2+\left(y-\frac{d\sqrt{3}}{2}\right)^2=d^2\{d=a\}\{0\le x\le c_1\}\{y\le d\}$$

<u>Inflating a square</u>

Let us take the square with coordinates A(0,0), B(0, l), C(l, l) and D(l, 0) => d = l√2. The equations for the sides of the square are: $x=0\{0\le y\le l\}$ $y=0\{0\le x\le l\}$

7 $x=l\{0\le y\le l\}$ 9 $y=l\{0\le x\le l\}$




Step 1 – trace the circle arch from A, radius = diagonal (arch CE), trace the circle arch from D, radius = diagonal (arch BE), take their intersection => E - no longer a symmetrical figure $(x-l)^2+y^2=2l^2\left\{0\le x\le\frac{l}{2}\right\}\{y\ge0\}$ $\left(\frac{l}{2},\frac{l\sqrt{7}}{2}\right)$ Label: E

$$x^2+y^2=2l^2\left\{\frac{l}{2}\le y\le\frac{l\sqrt{7}}{2}\right\}\{x\ge0\}$$

Step 2 – trace arch AD from E – the radius d = the diagonal of the square

$$\left(x-\frac{l}{2}\right)^2+\left(y-\frac{l\sqrt{7}}{2}\right)^2=2l^2\{y\le0\}$$

Step 3 - repeat Step 1 from A and B, obtaining F $\left(\frac{l\sqrt{7}}{2},\frac{l}{2}\right)$ Label: F

Step 4 - repeat Step 2 from F, obtaining arch AB

19 $$\left(x-\frac{l\sqrt{7}}{2}\right)^2+\left(y-\frac{l}{2}\right)^2=2l^2\{x\le0\}$$