Cet article est rédigé par des élèves. Il peut comporter des oublis et imperfections, autant que possible signalés par nos relecteurs dans les notes d'édition.

Pizza sharing

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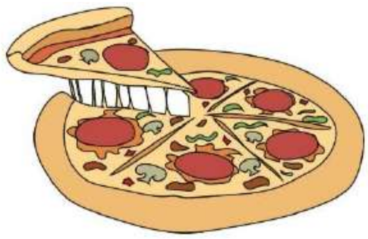
Abstract

Our task deals with being able to calculate how big a pizza is starting from a single edge of a slice and figuring out what tools are needed to do this.

We also need to figure out how to find the centre of a pizza and how to split it in half by using different tools.



The problem

When you get home, you find that your brother has eaten almost the entire pizza.

A) He spared just an edge of a slice for you. Starting from that edge, can you figure out how big that pizza was?

What tools do you need to calculate the exact diameter of the pizza?

You buy a new pizza just for you and your sister.

B) Can you find the centre of the pizza using a straightedge and/or a compass?

C) The same question if you are using just a sharp knife.

D) Using just the sharp knife, can you cut the pizza equally, for you and your sister, without using the centre of the pizza?

Solution for A

1. Starting from the remaining pizza edge we measure the chord AB using a ruler (a chord is a straight line segment that connects any two points on the circumference of the circle);

2. With a ruler we calculate the midpoint of AB and we take a perpendicular line using a triangle ruler to the arc itself, call it S (sagitta), and measure it.

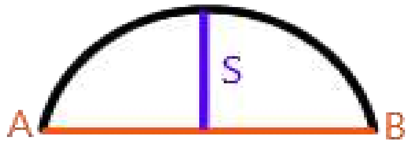


Figure 1. *The construction of sagitta and chord AB*

We need to calculate the radius (R) of the arc which will be the radius of the circle of which it is a part, and then the diameter of the circle.

Notations:

AB – chord
S – sagitta (hight of arch)
R – radius of the circle
O – centre of the circle

Applied mathematical theory:

We know that $AB/2$ and $R - S$ are two sides of a right triangle, with R the hypotenuse, and by using the Pythagorean theorem we will get:

$$R^2 = \left(\frac{AB}{2}\right)^2 + (R - S)^2$$

from which results in the following equations:

$$R = \frac{S^2 + \left(\frac{AB}{2}\right)^2}{2s} = \frac{S}{2} + \frac{AB^2}{8s}$$

The tools needed to calculate the exact diameter of the pizza: a ruler to measure the length of AB and S, a triangle ruler, a pencil, and paper.

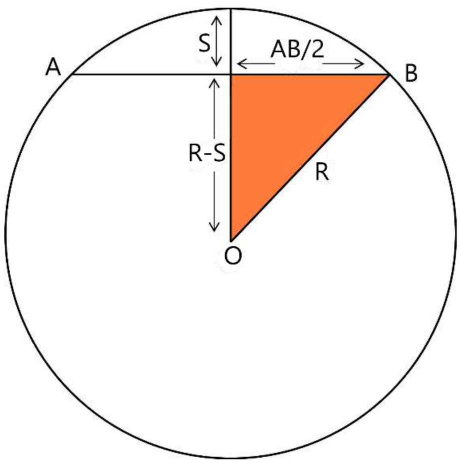


Figure 2*. The construction of the right triangle with R the hypotenuse*

Examples in practice:

- $S = 6$ cm $AB = 28$ cm

$$R = \frac{6}{2} + \frac{28^2}{8 \times 6} = 19{,}3\ cm$$

Diametre of pizza is 2R = 19.3x 2 = 38,6 cm

- $S = 6$ cm $AB = 36$ cm

$$R = \frac{6}{2} + \frac{36^2}{8 \times 6} = 30\ cm$$

Diametre of pizza is 2R = 30 x 2 = 60 cm

Solution for B

<u>Task: Find the centre of the pizza using a straightedge and/or a compass</u>

1. First we use a straightedge to draw a line segment inside the circle and label it AB - the exact location and size of the line are not important.

2. Using the compass we draw a semicircle with the centre in A.

3. Then we draw another semicircle by using the compass with the same opening size, but this time with the centre in B.

Note: The compass opening must be at least half the length of AB (so that the circles intersect each other).

4. We use a straightedge to draw a line segment through the points at which the two semicircles intersect (in the overlap space) and we extend the segment until crosses the rim of the original circle then label it as CD.

<u>The line CD is perpendicular to AB, bisects the line AB so it marks a diameter of our circle.</u>

5. Using a compass we will repeat the process that we did with A and B by drawing semicircles, one with the centre in C and one with the centre in D; these semicircles must overlap like a Venn diagram.
6. With a straightedge we draw a line segment through the points where the semicircles intersect and we extend until crosses the rim of the circle, label the new line EF.

<u>The line EF is perpendicular to CD, bisects the line CD so it marks a diameter of our circle.</u>

7. The crossing point of CD and EF is the centre of the circle - O marking the centre of the pizza.

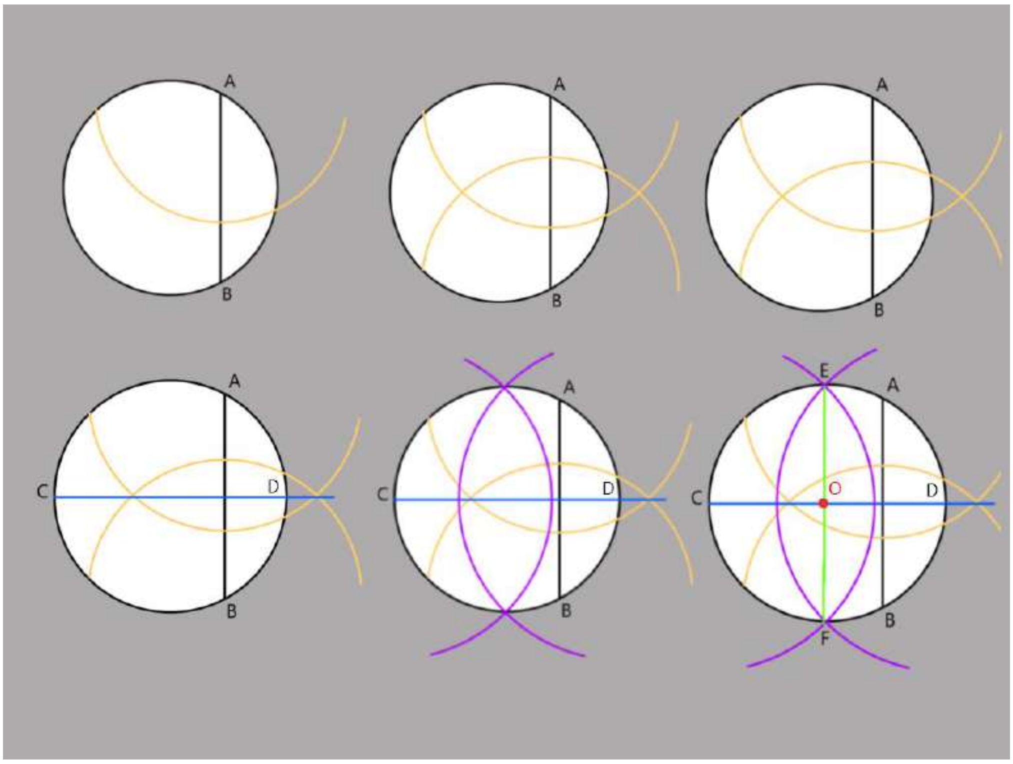


Figure 3. *Finding the centre of the pizza with a straightedge and a compass*

Solution for C

<u>Task: Find the centre of the pizza using a sharp knife.</u>

According to Jacob Steiner's theorem: It is **impossible** to find the centre of a given circle using only an unmarked straight line. However, with pizza, it's a little bit different.
Why?!… Because a pizza, although circular, is not a circle, and the big difference between straight lines on a circle and a knife cutting on a pizza is that we can move any pieces we cut out of the pizza and then match them in other locations to determine identical pieces.

On the other hand, using only straight lines on a circle cannot create duplicate angles.
In **Figure 4**, the piece drawn in pink can be removed and used to reproduce that particular angle around the centre, subtended by it.

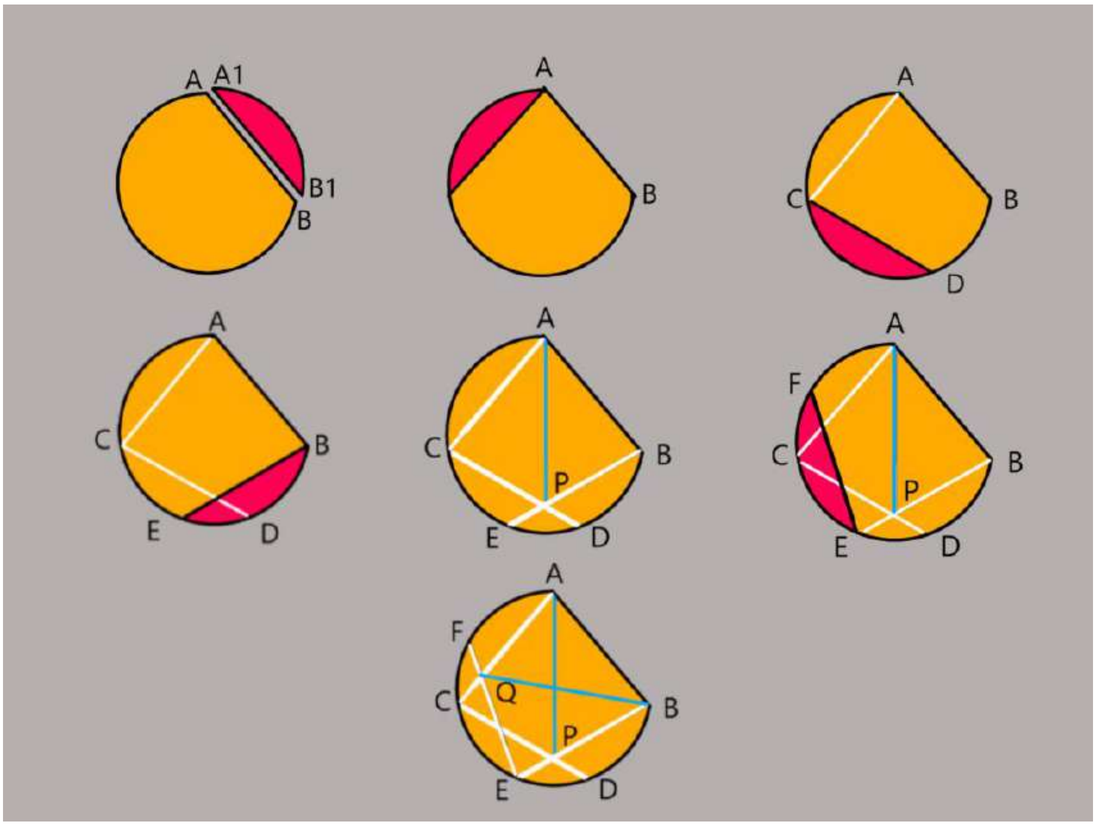


Figure 4. *Finding the centre of the pizza using a sharp knife, step by step*

Step 1: We cut a piece of the circle whose length we approximate is greater than a quarter of the entire circle ($A1B1$). It implies an angle at the centre bigger than 90°;
Step 2: We use the cut part to determine chord $AC;$
Step 3: Using the piece $A1B1$ we continue building the chord CD;
Step 4: Using the piece $A1B1$ we build the chord BE;
Step 5: We use the knife to mark the segment AP;
Step 6: We use the piece $A1B1$ to build the chord EF;
Step 7: We use the knife to mark the segment BQ.

The intersection of BQ with AP is the centre of the circle - O.

Proof:

To demonstrate that O is the centre of the pizza, we have to prove that $Q'B$ and AP' are diameters and the point they intersect is the centre of the circle.
Firstly, to show that AP' is a diameter, we mark the intersection of AP with the circle as P'.
We know that $AC = AB$ so the arches $\widehat{AB}$ and $\widehat{AC}$ are congruent.
We know that $CD = BE$ so the arches $\widehat{CD}$ and $\widehat{BE}$ are congruent.
$\widehat{ED}$ is in both arches $\widehat{CD}$ and $\widehat{BE}$, therefore the arches $\widehat{CE}$ and $\widehat{DB}$ are congruent.
In the triangle ΔABC, $AB = AC \Rightarrow \Delta ABC$ isosceles, AP bisects the angle $BAC \Rightarrow P'$ is the middle point of the arch $\widehat{CB}$ so the arches $\widehat{CE}$ and $\widehat{DB}$ are congruent => $\widehat{EP'}$ and $\widehat{P'D}$ are congruent.

So, $\widehat{AC}$ +$\widehat{CE}$ +$\widehat{EP'}$ =$\widehat{P'D}$ +$\widehat{BD}$ +$\widehat{BA}$ ⇒ arches $\widehat{ACP'}$ =$\widehat{P'BA}$ => AP' is a diameter.

The same, we prove that BQ' is a diameter in the circle.

We conclude that the intersection points of BQ and AP is the centre of the pizza.

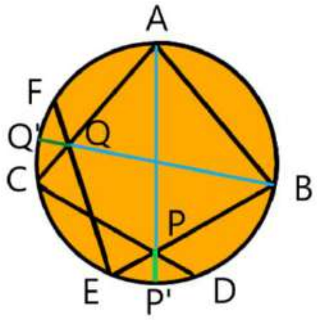


Figure 5. *Demonstration of finding the centre of the pizza using a sharp knife*

Solution for D

<u>Task: Using just the sharp knife, can you cut the pizza equally, for you and your sister, without using the centre of the pizza?</u>

Pizza Theorem says as follows:
"If a circular pizza is divided into 8, 12, 16, ... slices by making cuts at equal angles from an arbitrary point, then the sums of the areas of alternate slices are equal."

So, for sharing a pizza with another person, there's no need to cut it into precisely equal slices. Through an arbitrary point inside the circle using a pizza knife we cut 4 straight by rotating the knife at equal angles (with 45°) between each other regardless of the point of intersection, or the angle of the first cut (**Figure 6**).

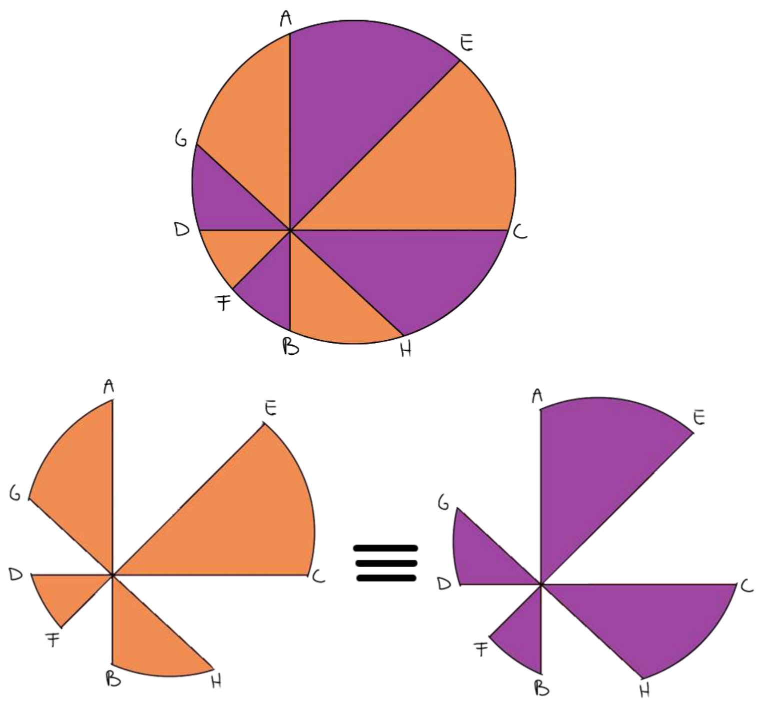


Figure 6. *Knife-slicing modes on a pizza*

If we take alternate slices identically coloured, the two portions will have the same amount of pizza in the measured area.
So, the pizza has been equally divided, and also the crust!

This theorem is true if the number of slices is any multiple of 4 except for 4, and the slices are cut by using equal angles through a fixed arbitrary point in the pizza.

Proof: Larry Carter and Stan Wagon came up with this "proof without words": in 1994 for The *Pizza Theorem.*
In mathematics, a proof without words is an illustration of an identity or mathematical statement which can be demonstrated as self-evident by a diagram without any accompanying explanatory text. The theorem is more related to circular geometry and the proofs have a lot of algebra manipulation of area of a circle and area sector.

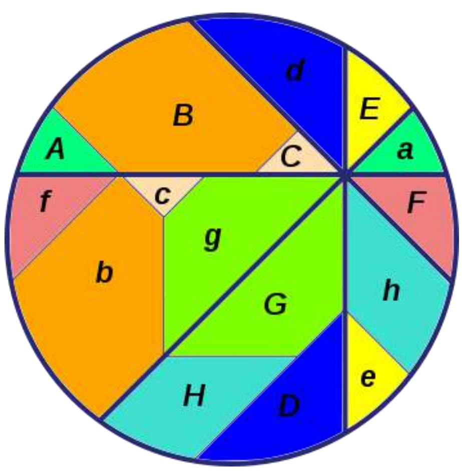


Figure 7. *Proof without words Carter and Wagon 1994*

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- *"Radius - Definition and More from the Free Merriam-Webster Dictionary". Merriam-webster.com. Retrieved 2012-05-22.*