A Comprehensive Study on Koch's Flake

A Fractal with Infinite Perimeter and Finite Area

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Structure of the Paper

1. **Introduction**: We present the central problem of our study, which is the paradoxical nature of fractals having an infinite perimeter but a finite area.
2. **Announcement of Conjectures and Results Obtained**: We state our primary conjecture about the properties of Koch's Flake and summarize the results of our investigation.
3. **Fractal Construction - The von Koch Flake**: We describe the geometric construction of the Koch Snowflake and explain the variables used in our demonstration.
4. **Proof of the Infinite Perimeter of the Flake**: We provide a mathematical proof that the perimeter of the Koch Snowflake is infinite.
5. **Proof of a Finite Area of the Flake**: We demonstrate that despite its infinite perimeter, the area of the Koch Snowflake is finite.
6. **Conclusion**: We summarize our findings and discuss the implications of our study.

1. Presentation of the Subject

The central problem we explore in this study is a thought-provoking question in geometry: Can a shape exist that has an infinite perimeter but a finite area? This question, at first glance, appears to contradict our basic understanding of geometric principles. The concept of a shape with an infinite boundary yet confined within a finite space challenge our conventional wisdom about the nature of geometric figures.

To address this problem, we turn to the realm of fractals. Fractals are mathematical constructs that exhibit unique properties. Some fractals, such as the Koch Snowflake, have an infinite perimeter while their area remains finite. This property emerges as we iteratively add smaller and smaller shapes to the initial fractal figure. With each iteration, the boundary of the fractal extends indefinitely, while the total area, though increasing, never surpasses a certain limit. This is due to the fact that the added shapes decrease in size with each iteration, contributing progressively less to the total area.

In essence, the problem we are addressing is understanding the nature and properties of fractals, specifically how they can exhibit the paradoxical characteristic of having an infinite perimeter and a finite area.

2. Announcement of Conjectures and Results Obtained

Our mathematical investigation into the properties of Koch's Flake, a specific type of fractal, yielded significant results. The primary conjecture was that despite having an infinite perimeter, the area of Koch's Flake remains finite.

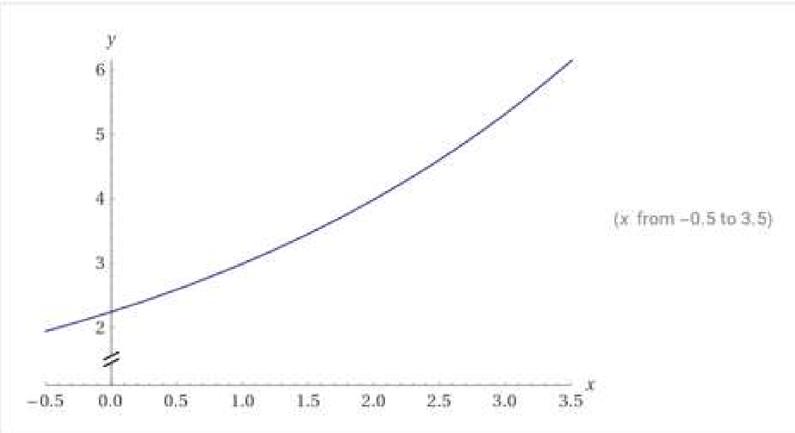
The Koch flake is a fractal constructed using an iterative process, so we can establish a specific notation with P_n, the perimeter at stage n of construction.

Our results demonstrate that as the number of iterations approaches infinity, the perimeter of Koch's Flake also approaches infinity. Mathematically, this is represented as:

$$\lim_{n \to \infty} P_n = +\infty$$

This result confirms that the perimeter of Koch's Flake is indeed infinite. The graphical representation of this result is given by the function (1):

$$P(x) = 3 \times \left(\frac{4}{3}\right)^{x-1}$$



Graphical visualisation of the function P, representative of the perimeter of the fractal.

We can observe that the function

$$P(x) = 3 \times \left(\frac{4}{3}\right)^{x-1}$$

seems to be an exponential function.

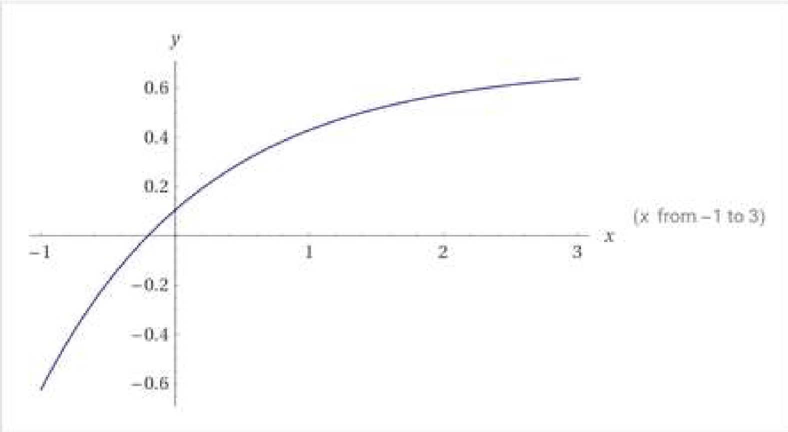
Simultaneously, we found that the area of Koch's Flake, although increasing at each iteration, converges to a finite value.

To do this, we first calculated the area A_n at stage n of the flake, and then determined its limit:

$$A(n) = \frac{\sqrt{3}}{4} + \frac{3\sqrt{3}}{20}\left(1 - \left(\frac{4}{9}\right)^{n-1}\right)$$

This shows that the sequence of areas can be bounded by $\frac{2\sqrt{3}}{5}$, indicating that the area of Koch's Flake is indeed finite:

$$\lim_{n\to\infty} A_n = \frac{2\sqrt{3}}{5}$$



Graphical visualisation of the function A, representative of the area of the fractal.

We can observe that the function:

$$A(x) = \frac{\sqrt{3}}{4} + \frac{3\sqrt{3}}{20}\left(1 - \left(\frac{4}{9}\right)^{x-1}\right)$$

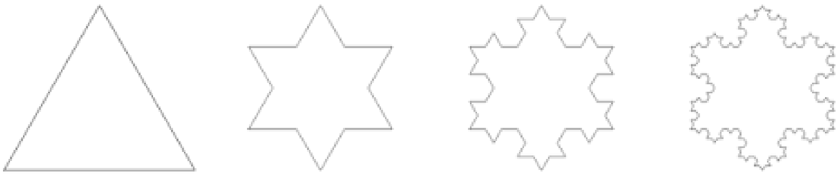
seems to be a decreasing exponential function.

In conclusion, our results provide mathematical proof that a geometric figure like Koch's Flake can indeed possess an infinite perimeter while maintaining a finite area. This might seem paradoxical, but it is theoretically possible and has been confirmed through our rigorous mathematical investigation.

3. Fractal construction - The von Koch flake

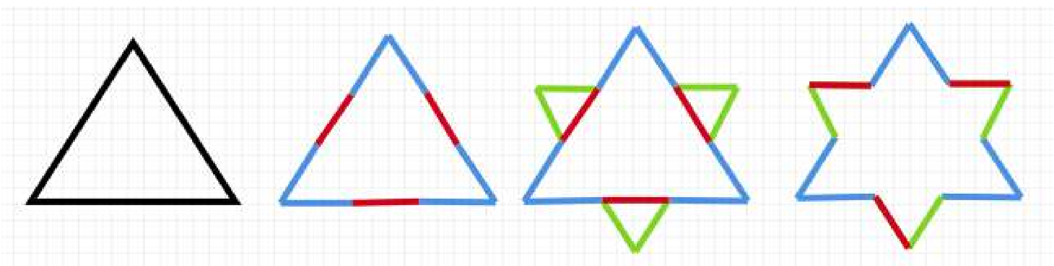
A. Geometric Construction

The von Koch flake, also known as the Koch snowflake, is a mathematical curve and one of the earliest fractal curves to have been described. It is named after the Swedish mathematician Helge von Koch, who introduced this geometric figure in a 1904 paper.



The first 4 steps of the fractal

At each stage of constructing this geometric figure, every segment is divided into three equal parts. An additional equilateral triangle is then formed between the two outer segments, effectively transforming the original segment into four new ones.



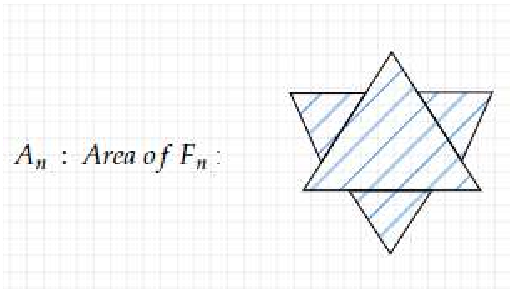
This addition of a smaller equilateral triangle, which has side lengths equal to one-third of the initial side length, results in two new sides (shown in green). The third side of the smaller triangle merges with the original side of the larger triangle (shown in red). However, it's important to note that the middle segment from the original division is removed (shown in red), and only the green side is added to the perimeter value.

B. Variables used in the demonstration

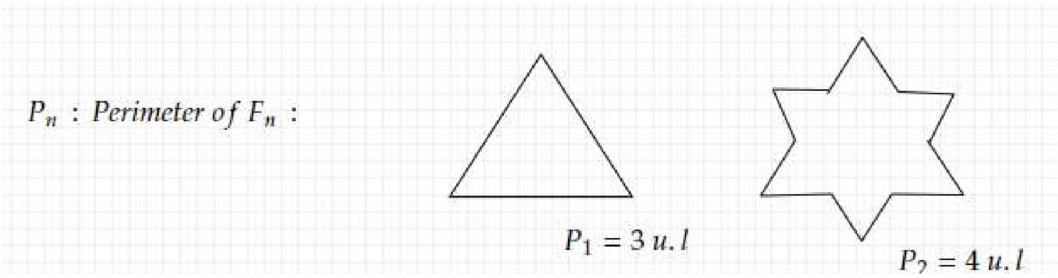
We assume the length of the first triangle is 1 unit length ($u.l$).

With n, the number of steps:

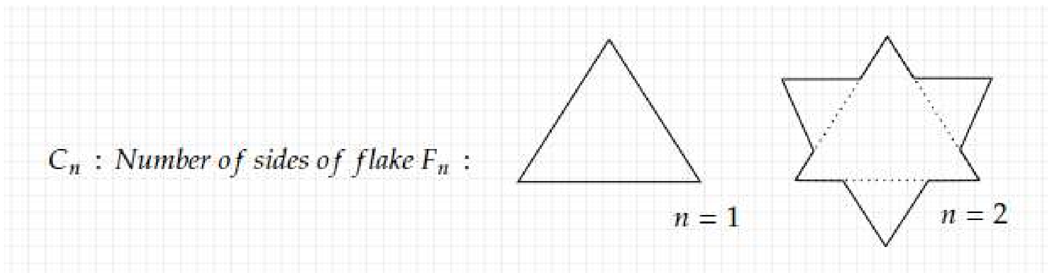
- A_n **: Area of** F_n



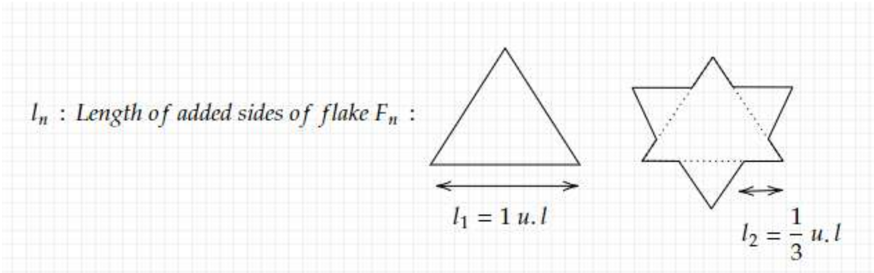
- P_n **: Perimeter of** F_n



- C_n **: Number of sides of flake** F_n



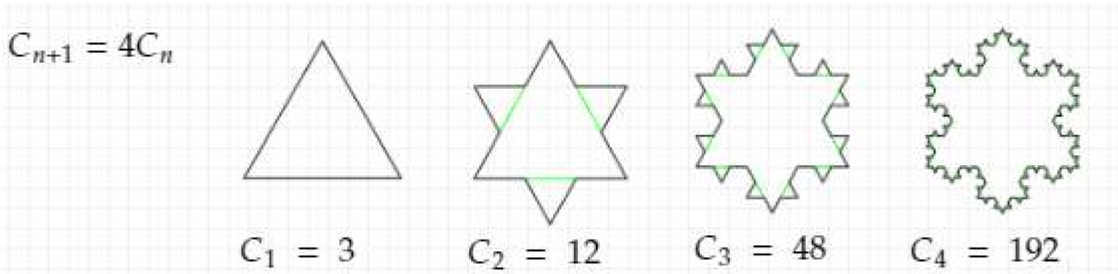
- l_n **: Length of added sides of flake** F_n



4. Proof of the Infinite Perimeter of the Flake (2)

A. Logical Evolution of the Flake's Side Number Based on n

Throughout our analysis, n belongs to the set of natural numbers, $\mathbb{N}$. The geometric figure representing the flake, denoted as F_1, has three sides of a triangle, hence $C_1 = 3$.



At each step, each segment of the geometric object is divided into 3 equal segments, which multiplies the number of segments by 4.

Each iteration the number of sides is multiplied by 4: we have, for all $n \geq 1$:

$$C_{n+1} = 4C_n$$

The sequence C_n is geometric with common ratio $q = 4$ and a first term $C_1 = 3$, we find the explicit formula

$$\forall n \geq 1, \quad C_n = C_1 \times q^{n-1} = 3 \times 4^{n-1}$$

B. Evolution of Each Side Length of F_n

Due to the fractal's unique construction, each segment of F_n has a length equal to l_n.

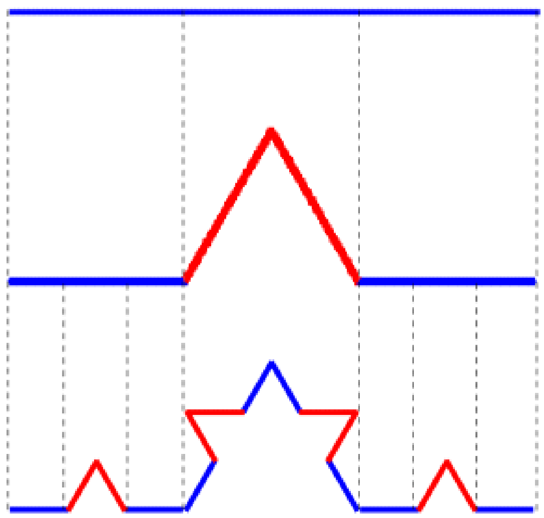


Illustration of the evolution of one side of the fractal according to the first 3 stages

Let's denote $l_1 = 1$.

At each iteration, the side length is divided by three, hence for all $n \geq 1$, $l_{n+1} = \frac{1}{3} l_n$.

This is also a geometric sequence with a common ratio $q = \frac{1}{3}$ and an initial term $l_1 = 1$ u.l. . Using the same process as before, we can derive the explicit formula:

$$\forall n \geq 1, \quad l_n = l_1 \times q^{n-1} = \left(\frac{1}{3}\right)^{n-1}$$

C. <u>Derivation of Perimeter Expression</u>

To find the perimeter formula, we need to multiply the two sequences (C_n) and (l_n) .

Hence, $P_n = C_n \times l_n = 3 \times 4^{n-1} \times \left(\frac{1}{3}\right)^{n-1} = 3 \times \left(\frac{4}{3}\right)^{n-1}$

By deduction, we find

$$P_{n+1} = 3 \times \left(\frac{4}{3}\right)^{\mathrm{n}} = 3 \times \left(\frac{4}{3}\right)^{\mathrm{n}-1} \times \frac{4}{3} = \frac{4}{3} P_n$$

This perimeter expression is consistent with our observations from the initial steps. Indeed, we have

$$P_n = 3 \times \left(\frac{4}{3}\right)^{n-1}$$

This relation confirms that (P_n) is a geometric sequence with a common ratio $q = \frac{4}{3}$ and an initial term $P_1 = C_1 \times l_1 = 3$. Hence, $P_{n+1} = \frac{4}{3} P_n$.

Since $\frac{4}{3} > 1$, this sequence diverges, and we have:

$$\lim_{n \to \infty} P_n = +\infty$$

The limit of P_n as n approaches infinity is $+\infty$. Therefore, we can conclude that the perimeter of the flake is infinite.

5. Proof of a finite area of the flake (3)

We will now attempt to prove that the area of this flake is finite.

Let's start by finding the area of the initial triangle, which we will denote as A_1. We know that F_n is an equilateral triangle, so we have:

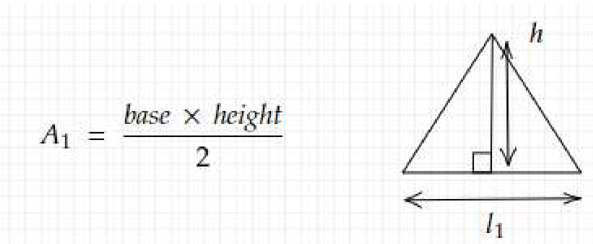
$$A_1 = \frac{1}{2} \times \ base \ \times \ height = \frac{1}{2} \times l_1 \times h$$

where h is the height of the triangle, which can be found using the Pythagorean theorem:

$$h^2 = {l_1}^2 - \left(\frac{l_1}{2}\right)^2 \Longrightarrow \mathrm{h} = l_1 \times \frac{\sqrt{3}}{2}$$

Substituting h into the formula for A_1, we get

$$A_1 = \frac{1}{2} \times l_1 \times l_1 \times \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4}$$



Given that we have chosen $l_1 = 1$u.l as our starting length, we have $A_1 = \frac{\sqrt{3}}{4}$

By deduction, $A_n = \frac{l_{n+1}}{2} \times l_{n+1} \frac{\sqrt{3}}{2}$

We observe that when we have F_n, we construct F_{n+1} by adding an equilateral triangle to each side of F_n with a side length of l_{n+1}. Mathematically, for each $n \in \mathbb{N}$ and $n \geq 1$, we have

$$A_{n+1} = A_n + C_n \left(\frac{1}{2} l_{n+1} \times l_{n+1} \frac{\sqrt{3}}{2}\right)$$

After some development, we obtain the following result:

$$A_{n+1} = A_n + C_n \left(\frac{1}{2} l_{n+1} \times l_{n+1} \frac{\sqrt{3}}{2}\right)$$

$$A_{n+1} = A_n + \frac{\sqrt{3}}{4} \times C_n \times l_{n+1}^2$$

$$A_{n+1} = A_n + \frac{\sqrt{3}}{4} \times 3 \times 4^{n-1} \times \left(\frac{1}{3} l_n\right)^2 \qquad with : l_{n+1} = \frac{1}{3} l_n$$

$$A_{n+1} = A_n + \frac{\sqrt{3}}{4} \times 3 \times 4^{n-1} \times \left(\frac{1}{3} \times \frac{1}{3^{n-1}}\right)^2 \qquad with\ :\ ln = \left(\frac{1}{3}\right)^{n-1}$$

$$A_{n+1} = A_n + \frac{\sqrt{3}}{4} \times 3 \times 4^{n-1} \times \left(\frac{1}{3} \times \frac{1}{3^n} \times \frac{1}{3^{-1}}\right)^2$$

$$A_{n+1} = A_n + \frac{\sqrt{3}}{4} \times 3 \times 4^{n-1} \times \left(\frac{1}{3} \times \frac{1}{3^n} \times 3\right)^2$$

$$A_{n+1} = A_n + \frac{\sqrt{3}}{4} \times 3 \times 4^{n-1} \times \frac{1}{3^{2n}}$$

$$A_{n+1} = A_n + \frac{3\sqrt{3}}{4} \times \frac{1}{4} \times 4^n \times \frac{1}{3^{2n}}$$

$$A_{n+1} = A_n + \frac{3\sqrt{3}}{16} \times \frac{4^n}{3^{2n}}$$

Which brings us to the result:

$$A_{n+1} = A_n + \frac{3\sqrt{3}}{16} \times \left(\frac{4}{9}\right)^n$$

<u>Another way of calculating the area:</u>

Another perspective to calculate the area of the Koch Snowflake involves considering the addition of new triangles at each iteration. In each step, a new triangle is added to each side of the previous iteration. The side length of these new triangles is one third of the side length of the triangles from the previous iteration.

Therefore, the area of a new triangle will be $\frac{1}{3^2} = \frac{1}{9}$ of the area of a triangle from the previous iteration.

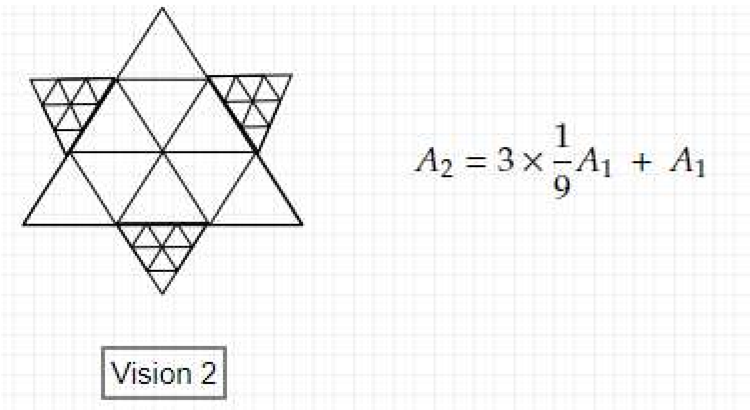


Illustration of the second way of calculating the area of the flake

Let's denote M_n as the area of a new triangle at the n-th iteration.

We can then express the total area of the snowflake at the nth iteration, A_{n+1}, as the sum of the area of the snowflake at the previous iteration, A_n, and the total area of all new triangles added at the nth iteration, which is $M_n \times C_n$, where C_n is the number of sides (and thus the number of new triangles added) at the n-th iteration:

$$\mathrm{A_{n+1}} = \mathrm{A_n} + \mathrm{M_n} \times \mathrm{C_{n,}}$$

Given the fractal nature of the Koch Snowflake, the area of a new triangle at the nth iteration, M_{n+1}, is $\frac{1}{9}$ of the area of a triangle from the previous iteration A_n . Therefore, we can express M_n as

$$M_n = \left(\frac{1}{9}\right)^n \times A_1$$

Substituting this into our equation for A_{n+1} gives:

$$A_{n+1} - A_n = \left(\frac{1}{9}\right)^n \times \frac{\sqrt{3}}{4} \times 3 \times 4^{n-1}$$

$$A_{n+1} - A_n = \frac{1^n}{9^n} \times 4^n \times \frac{\sqrt{3}}{4} \times 3 \times \frac{1}{4}$$

$$A_{n+1} - A_n = \frac{4^n}{9^n} \times \frac{3\sqrt{3}}{4} \times \frac{1}{4}$$

$$A_{n+1} - A_n = \boxed{\left(\frac{4}{9}\right)^n \times \frac{3\sqrt{3}}{16}}$$

This result is consistent with our previous calculation, confirming the validity of this approach.

Thus, the area of the Koch Snowflake, denoted as A_n, is an increasing sequence, as $\left(\frac{4}{9}\right)^n \times \frac{3\sqrt{3}}{16} > 0$ for all $n \in [1, +\infty)$.

(We will now see that the area does not increase indefinitely but converges to a finite value (4))

We are interested in studying the variation of this sequence and determining whether it converges or diverges.

This sequence is neither arithmetic nor geometric, so we will examine the difference between consecutive terms, A_{n+1} *and* A_n.

Let's denote k as an integer.

It's clear that for all $k > 1$, $A_{k+1} - A_k = \frac{3\sqrt{3}}{16} \times \left(\frac{4}{9}\right)^k$. At first glance, the variation of this sequence is not immediately apparent.

However, we can gain insight by considering the sum of all terms $A_{k+1} - A_k$, which will reveal the evolution of the area of the Koch Snowflake, F_n:

Since $A_n = \sum_{k=1}^{n-1}(A_{k+1} - A_k)$, we have

$$\sum_{k=1}^{n-1} (A_{k+1} - A_k) = \sum_{k=1}^{n-1} \left(\frac{3\sqrt{3}}{16} \times \left(\frac{4}{9}\right)^k \right)$$

$$= \frac{3\sqrt{3}}{16} \times \sum_{k=1}^{n-1} \left(\frac{4}{9}\right)^k$$

We can observe that the sum of $A_{k+1} - A_k$ for $k = 1$ to $k = 3$ simplifies to $A_{3+1} - A_1$, due to the telescoping nature of the sum.

This concept can be generalized to:

$$\sum_{k=1}^{3} (A_{k+1} - A_k) = \underset{k=1}{(A_2 - A_1)} + \underset{k=2}{(A_3 - A_2)} + \underset{k=3}{(A_4 - A_3)} = A_{3+1} - A_1$$

$$\sum_{k=1}^{3-1} (A_{k+1} - A_k) = \underset{k=1}{(A_2 - A_1)} + \underset{k=2}{(A_3 - A_2)} = A_3 - A_1$$

Substituting this into our equation gives:

$$A_n - A_1 = \frac{3\sqrt{3}}{16} \times \sum_{k=1}^{n-1} \left(\frac{4}{9}\right)^k$$

We can then solve for A_n:

$$
\begin{aligned}
A_n &= A_1 + \frac{3\sqrt{3}}{16}\sum_{k=1}^{n-1}\left(\frac{4}{9}\right)^k \\
&= \frac{\sqrt{3}}{4} + \frac{3\sqrt{3}}{16}\sum_{k=1}^{n-1}\left(\frac{4}{9}\right)^k \\
&= \frac{\sqrt{3}}{4} + \frac{3\sqrt{3}}{16}\left(\frac{1-\left(\frac{4}{9}\right)^n}{1-\frac{4}{9}}\right) \qquad \textit{using } \sum_{k=0}^{n} q^k = \frac{1-q^{n+1}}{1-q} \textit{ with } q \neq 1 \\
&= \frac{\sqrt{3}}{4} + \frac{3\sqrt{3}}{16}\left(\frac{1-\left(\frac{4}{9}\right)^{n-1}\times\frac{4}{9}}{\frac{5}{9}}\right) \\
&= \frac{\sqrt{3}}{4} + \frac{3\sqrt{3}}{16}\left(\frac{1-\left(\frac{4}{9}\right)^{n-1}}{\frac{5}{9}}\times\frac{4}{9}\right) \\
&= \frac{\sqrt{3}}{4} + \frac{3\sqrt{3}}{16}\left(\frac{4}{9}\times\frac{1-\left(\frac{4}{9}\right)^{n-1}}{\frac{5}{9}}\right) \\
&= \frac{\sqrt{3}}{4} + \frac{3\sqrt{3}}{16}\left(\frac{4}{9}\times\frac{9}{5}\times\left(1-\left(\frac{4}{9}\right)^{n-1}\right)\right) \\
&= \frac{\sqrt{3}}{4} + \frac{3\sqrt{3}}{16}\times\frac{4}{5}\left(1-\left(\frac{4}{9}\right)^{n-1}\right)
\end{aligned}
$$

And finally, we obtain:

$$\boxed{A_n = \frac{\sqrt{3}}{4} + \frac{3\sqrt{3}}{20}\left(1-\left(\frac{4}{9}\right)^{n-1}\right)}$$

As n approaches infinity, $\left(\frac{4}{9}\right)^{n-1}$ approaches 0, and thus $1-\left(\frac{4}{9}\right)^{n-1}$ approaches 1.

$$\lim_{n \to +\infty} \left(\frac{4}{9}\right)^{n-1} = 0 \Leftrightarrow \lim_{n \to +\infty} 1 - \left(\frac{4}{9}\right)^{n-1} = 1 \qquad \text{because } -1 < \frac{4}{9} < 1$$

And

$$\lim_{n \to +\infty} \frac{3\sqrt{3}}{20} \left(1 - \left(\frac{4}{9}\right)^{n-1}\right) = \frac{3\sqrt{3}}{20}$$

Therefore, using the propriety of the addition of two limits of the sequences, we conclude that the limit of the sequence A_n as n approaches infinity is $\frac{\sqrt{3}}{4} + \frac{3\sqrt{3}}{20} = \frac{2\sqrt{3}}{5}$.

This shows that the sequence A_n can be bounded by $\frac{2\sqrt{3}}{5}$.

In conclusion, we have demonstrated that the Koch Snowflake, a specific type of fractal, can have an infinite perimeter while possessing a finite area. This may seem paradoxical, but it is theoretically possible.

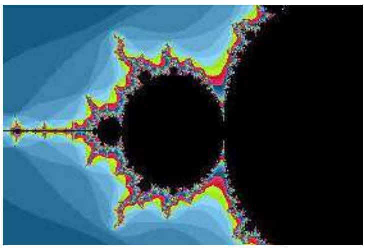
6. Conclusion

In conclusion, the study of fractals, such as the Koch Snowflake, has profound implications across a variety of fields. Their unique properties, including infinite complexity and self-similarity, provide valuable insights for enhancing industrial processes, computer graphics, and understanding material characteristics.

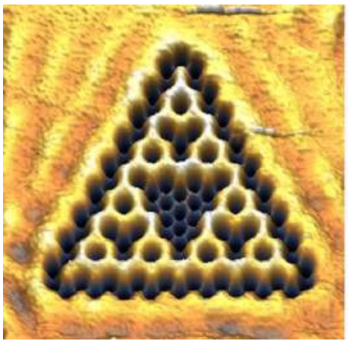
The Koch Snowflake, in particular, is a fascinating example of a shape with finite area but infinite perimeter. This seemingly paradoxical property challenges our traditional understanding of dimensions and opens up new avenues for exploration in mathematics and physics.

Moreover, the Koch Snowflake's geometric progression and self-similar structure have been used as a model in various scientific fields, from computer graphics to the study of natural phenomena like snowflakes and coastlines.





In the realm of quantum physics, the concept of fractals has been applied in the creation of a quantum fractal, a significant breakthrough that opens up new possibilities for the application of fractals in quantum physics.



Source : ***"Physicists wrangled electrons into a quantum fractal" By [Emily Conover](#) (https://www.sciencenews.org/article/physicists-wrangled-electrons-quantum-fractal,)***

In essence, the study of the Koch Snowflake and other fractals is not just a mathematical curiosity, but a key to unlocking advancements in various fields. As we continue to delve deeper into the world of fractals, we can expect to uncover more of their potential applications and contributions to science and industry.

7. Acknowledgements

We would like to express our deepest gratitude to our supervisor, Sébastien CASTAGNEDOLI, and Elodie FORT for their continuous support and invaluable guidance throughout the course of this research. We also thank our researcher, Serge DUMONT, for his insightful comments and suggestions. Our sincere appreciation goes to Emmanuel D'Alzon High School Nîmes for providing us with the necessary resources and environment to conduct this study. Lastly, we are grateful to our peers and the proofreaders for their constructive feedback and encouragement.

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Notes d'édition

(1) The result proved here is that, after n steps in the construction of Koch snowflake, the perimeter is $P_n = 3\left(\frac{4}{3}\right)^{n-1}$; hence the function $P(x)$ is very useful to give a graphic representation of the growth of the perimeter.

(2) Proof that the perimeter of the flake is infinite.

(3) Proof that the area of the flake is finite.

(4) In fact, the area **does increase infinitely**, but goes to a finite value when n goes to infinity