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Homework Planning

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Abstract

In this article we consider a problem relating to the mathematical modelling of homework assignments. We develop recurrence relations to solve the problem and expand it by considering generalizations and obtaining general formulas for their solutions. One particular focus is on probabilistic variants of the original problem.

1. Presentation of the Research Topic

A university course consists of 14 lectures, and the professor gives homework to his students at each lecture. Each homework will be marked by the professor by either P (pass) or F (fail). In order to qualify for the final exam, a student must have

- at most one missing homework (out of 14);
- no three consecutive homework fails.

In how many ways can a student organize his homework to qualify for the final exam? What if the consecutive requirement is removed?

2. Brief Presentation of the Results Obtained

The solution to the original problem above will be presented in section 3.2.

Before that, we first generalize the problem in section 3.1 to obtain the necessary tools that will allow us to easily solve it. In particular, we consider what happens when the numbers 14 and 3 in the problem statement are replaced by general n and k.

It turns out that if we let $x_{n,k}$ denote the number of ways a student can organize his homework to n lectures such that he has no k consecutive Fs, then the $x_{n,k}$ satisfy a beautiful recurrence relation

$$x_{n,k} = x_{n-1,k} + x_{n-2,k} + \ldots + x_{n-k,k} \quad \text{for all } n > k,$$

which generalizes the famous Fibonacci recurrence relation

$$F_n = F_{n-1} + F_{n-2} \quad \text{for all } n \geq 2.$$

Thus, our generalization not only has the advantage of being more versatile, solving more than one specific problem, it also sheds light on the fascinating hidden structure of the original problem.

In section 3.3 we present the standard way of obtaining an explicit formula for the n^{th} Fibonacci number and use it to obtain an explicit formula for $x_{n,2}$.

Section 3.4 introduces a further generalization of our problem by introducing a probabilistic element. Specifically, we study what happens if the student is assigned a pass with probability p and a fail with probability $1-p$ independently for each homework assignment. As in the deterministic case, we prove a recurrence relation and then develop explicit formulas in the case $k=2$ in section 3.5.

Finally, in section 3.6 we consider the limiting case $n=\infty$, in which the student has an infinite number of homework assignments.

3. Solution and Generalizations

3.1. The General Case without Missing Homework

In order to solve our original problem, we will first study a more general case, which can then be applied to obtain our solution. This approach allows us to apply our results to a wider range of problems and reveals a beautiful hidden link between our problem and the Fibonacci numbers.

Therefore, we will start the proof by solving the following general problem:

Problem (General Case, No Missing Homework)**.** *Let n, k be positive natural numbers. A university course consists of n lectures, and the professor gives homework to his students at each lecture. Each homework will be marked by the professor by either P (pass) or F (fail). In order to qualify for the final exam, a student must have no k consecutive homework fails and no missing homework. If we denote by $x_{n,k}$ the number of ways a student can organize his homework to qualify for the final exam, determine $x_{n,k}$.*

We begin by analysing a few special cases for small k, before moving on to prove our main result. To ease notation, we will suppress the subscript k in $x_{n,k}$ in this section.

Case I: $k=1$

This means that the student should never be marked with F or, equivalently, he should always be marked with P.

Obviously in this situation, we only have one possibility, namely n consecutive passes. We have

$$x_n = 1 \quad \text{for all } n.$$

Case II: $k=2$

This means that if the student is marked with F, he has to be marked with P for his next homework.

First, we will analyse a few particular cases for n:

- $n=1$

  The student can be marked either with P or with F (in order to qualify for the final exam). Obviously this leads us to

  $$x_1 = 2.$$

- $n=2$

  The student can receive either 0 Fs or only one F for homework no. 1 or homework no. 2. This leads us to

  $$x_2 = 1+2 = 3.$$

- $n = 3$

  The student can receive

  - 0 Fs; or
  - only one F for homework no. 1, homework no. 2 or homework no. 3; or
  - 2 Fs for homework no. 1 and homework no. 3.

  This leads us to

  $$x_3 = 1 + 3 + 1 = 5.$$

- $n = 4$

  The student can receive

  - 0 Fs; or
  - only one F for homework no. 1, homework no. 2, homework no. 3 or homework no. 4; or
  - 2 Fs for homework
    - no. 1 and no. 3
    - no. 1 and no. 4
    - no. 2 and no. 4.

  This leads us to:

  $$x_4 = 1 + 4 + 3 = 8.$$

- $n = 5$

  The student can receive:

  - 0 Fs; or
  - only one F for homework no. 1, homework no. 2, ... or homework no. 5; or
  - 2 Fs for homework
    - no. 1 and no. 3
    - no. 1 and no. 4
    - no. 1 and no. 5
    - no. 2 and no. 4
    - no. 2 and no. 5
    - no. 3 and no. 5
  - 3 Fs (without being any 2 consecutive Fs) for homework no. 1, 2 and 3.

  This leads us to

  $$x_5 = 1 + 5 + 6 + 1 = 13.$$

As you can see, this case by case analysis is getting more and more difficult and involved for higher values of n, and it wouldn't be easy to find a 'by hand' solution for the case $n = 14$ needed for our original problem.

Therefore, we will try a different approach and we will try to determine the value of x_n not by an explicit formula but by a recurrence relation. This idea was inspired by noticing that $x_1 = 2$, $x_2 = 3$, $x_3 = 5$, $x_4 = 8$, $x_5 = 13$ is part of the famous *Fibonacci sequence*.

Definition (Fibonacci sequence)**.** *The Fibonacci sequence is defined by the following recurrence relation:*

$$F_n = F_{n-1} + F_{n-2} \text{ for all } n \geq 2$$

with initial conditions $F_0 = 1$, $F_1 = 1$.

We hence would like to prove that:

Proposition 3.1. *In the case $k = 2$, $(x_n)_{n\geq 1}$ satisfies the following recurrence relation:*

$$x_n = x_{n-1} + x_{n-2} \text{ for all } n > 2$$

with initial conditions $x_1 = 2$, $x_2 = 3$.

Proof. We consider two cases, based on the student's first homework outcome.

If we assume that the student's number 1 homework is marked with P, then for the next $n-1$ homework assignments the student finds himself in the $n-1$ case i.e. the student can receive for assignments 2 to n any arrangement of Ps and Fs that would be admissible if he had instead $n-1$ assignments in total. Thus, if homework number 1 is marked with P, then the number of ways in which the student can organize his homework to qualify for the final exam is x_{n-1}.

If we assume that the student's number 1 homework is marked with F, then it is necessary for him to receive P for his second homework. For the next $n-2$ homework assignments he has no further restrictions and so he finds himself in the $n-2$ case. Thus, if homework number 1 is marked with F, then the number of ways in which the student can organize his homework to qualify for the final exam is x_{n-2}.

In conclusion, taking into account the 2 situations above, we conclude

$$x_n = x_{n-1} + x_{n-2} \text{ for all } n \geq 3.$$

The initial conditions were determined in the case-by-case work above. □

We are now finally ready to tackle the case for general $k \geq 3$.

Case III: $k \geq 3$

We will take into consideration the following situations for n:

- $n < k$

  The student can receive for any homework either P or F, without any restriction, since it is impossible to receive k consecutive Fs. Therefore, in this case,

  $$x_n = 2^n,$$

  because x_n represents the number of functions from the set $\{1, 2, \dots n\}$ to the set $\{P, F\}$.

- $n = k$

  The only situation which the student has to avoid is to receive for all homework the mark F. Therefore, keeping in mind the $n < k$ case, we have

  $$x_n = 2^n - 1.$$

- $n > k$

  Similarly to the $k = 2$ case, we take into consideration the first k homework assignments. In order for the student to qualify for the final exam, at least one of the first k assignments has to be marked with a P. We denote by i the number of the first homework which is marked with a P. We must have $1 \leq i \leq k$. So, assignments 1 through $i-1$ are marked with an F, assignment i is marked with a P and then, for the next $n-i$ homework assignments, the student finds himself in the $n-i$ case, so he has x_{n-i} ways of receiving marks and qualifying for the final exam. Therefore, iterating over the different values of i we obtain that

  $$x_n = x_{n-1} + x_{n-2} + \ldots + x_{n-k}.$$

Taking into account the whole discussion, we have proved the following:

Proposition 3.2. *For $k \geq 1$ we have that $(x_{n,k})_{n\geq 1}$ satisfies the following recurrence relation:*

$$x_{n,k} = x_{n-1,k} + x_{n-2,k} + \ldots + x_{n-k,k}, \quad \textit{for all } n > k$$

with initial conditions

$$x_{n,k} = 2^n \textit{ for } 1 \leq n \leq k-1, \quad x_{k,k} = 2^k - 1.$$

3.2. Solving the Original Problem

We now have all the tools necessary to solve the original problem.

First, a discussion about how missing homework affects a chain of consecutive Fs is needed. We have chosen to assume that a missing homework does not reset the number of consecutive fails, as that would create a bad incentive for the student to miss a homework if he failed the last two. Hence, using our assumption, a sequence of the type $PFMFP$ is allowed, where M denotes a missing homework, while a sequence like $PFFMF$ is not because it contains three consecutive fails.

Since there are two questions, we begin by tackling the first one, namely that there are 14 assignments and the student qualifies for the final exam only if he does not have more than one missing homework and no more than two consecutive fails. We consider two cases:

- The student has a missing homework:

  The missing homework can be in one of the 14 places. Independently of which homework is missed, ignoring it, the student then has 13 homework assignments in which he must not have more than 2 consecutive failures. So, this case is equivalent to $n = 13$ and $k = 3$ in the general case. Hence, we must find $x_{13,3}$.

  Applying the recurrence formula obtained previously, we find that:

$$\begin{cases} x_{1,3} = 2 \\ x_{2,3} = 4 \\ x_{3,3} = 7 \\ x_{4,3} = 13 \\ x_{5,3} = 24 \\ x_{6,3} = 44 \\ x_{7,3} = 81 \\ x_{8,3} = 149 \\ x_{9,3} = 274 \\ x_{10,3} = 504 \\ x_{11,3} = 927 \\ x_{12,3} = 1705 \\ x_{13,3} = 3136 \end{cases}$$

  We obtain in this case $x_{13,3} \times 14 = 3136 \times 14 = 43904$ ways in which a student can qualify for the final exam.

- The student has no missing homework:

  This is equivalent to the general case with $n = 14$ and $k = 3$. So, in this case there are $x_{14,3} = x_{13,3} + x_{12,3} + x_{11,3} = 3136 + 1705 + 927 = 5768$ ways in which a student can qualify for the final exam.

Therefore, the student can qualify for the final exam in: $43904 + 5768 = 49672$ ways in total.

We now tackle the second part of the original problem, in which the consecutive requirement is removed. Again, we consider two different cases:

- The student has a missing homework:

  We can place the missing homework in 14 ways and then we have 13 other homework assignments to consider. We can only have 0, 1 or 2 fails.

  When we don't have any fails, we have only one possible arrangement, namely all passes.

  If there is one failed homework, there are 13 positions in which it can be placed.

  For two fails, the first has 13 possible positions in which it can be placed, and the second is left with 12. We will divide the product by two in order to count every combination only once, since the two fails are indistinguishable. Therefore, we obtain $1 + 13 + \frac{13\times 12}{2} = 92$ ways.

  We multiply the resulting sum by 14 to account for the placement of the missing homework. In conclusion, the student can qualify for the final exam in: $14 \times 92 = 1288$ ways.

- The student has no missing homework:

  Similarly to the previous case, we calculate the number of ways in which the failed homework assignments can be positioned, this time taking into consideration that we now have 14 positions to fill. We follow the same case by case approach to find that the student can qualify for the final exam in $1 + 14 + \frac{14\times 13}{2} = 106$ ways.

Therefore, in total we have $1288 + 106 = 1394$ ways in which the student can qualify for the final exam.

Thus, we have solved the original problem. Moreover, through Proposition 3.2 we have provided the tools to solve a wide range of similar problems.

3.3. Finding an Explicit Formula for $x_{n,2}$

In one of previous sections we observed that for $k = 2$ the recurrence formula for $(x_n)_{n\geq 1}$ coincides with that of the Fibonacci sequence (in this section, we again write x_n instead of $x_{n,2}$ for ease of notation). In fact, we have that $x_n = F_{n+2}$ for all $n \geq 1$. In what follows, we use the standard technique to obtain an explicit formula for F_n and use it to get an explicit formula for x_n.

So, subject to the following initial conditions:

$$\begin{cases} F_0 = 0 \\ F_1 = 1 \end{cases}$$

we look to solve:

$$F_{n+1} = F_n + F_{n-1} \quad \text{for all } \ n \geq 1. \tag{1}$$

We will try a solution of the type $F_n = s^n$ for all $n \geq 0$, for some $s \in \mathbb{R}^*$. Such a solution satisfies (1) if and only if

$$s^{n+1} = s^n + s^{n-1} \quad \text{for all } n \geq 1.$$

We divide by $s^{n-1} \neq 0$ and we obtain that (1) is satisfied if and only if $s^2 = s + 1$ i.e. $s^2 - s - 1 = 0$. The only solutions to this quadratic are $\phi := \frac{1+\sqrt{5}}{2}$ and $\psi := \frac{1-\sqrt{5}}{2}$.

Now, we try to combine the two solutions to (1) obtained above, $(\phi^n)_{n\geq 1}$ and $(\psi^n)_{n\geq 1}$, to fit the initial conditions (1). Hence we try a solution of the form: $F_n = A\phi^n + B\psi^n$.

We look for A and B such that the initial conditions are satisfied. We have:

$$\begin{cases} A+B=0 \\ A\phi+B\psi=1 \end{cases}$$
$$\Leftrightarrow \begin{cases} A=-B \\ B(\psi-\phi)=1 \end{cases}$$
$$\Leftrightarrow \begin{cases} A=-B \\ B\times(-\sqrt{5})=1 \end{cases}$$
$$\Leftrightarrow \begin{cases} A=\dfrac{1}{\sqrt{5}} \\ B=-\dfrac{1}{\sqrt{5}}. \end{cases}$$

So, now we have the general formula for F_n, for all $n \geq 1$:

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Since $x_n = F_{n+2}$ for all $n \geq 1$ we have:

$$x_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+2} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+2}.$$

3.4. Introducing Probabilities

We now introduce a new level of generality to our problem.

Problem (General Case with Probabilities). *Let n, k be positive natural numbers, $p \in [0,1]$. A university course consists of n lectures, and the professor gives homework to his students at each lecture. Each homework will be marked by the professor by either P (pass) or F (fail). In order to qualify for the final exam, a student must have no k consecutive homework fails and no missing homework. Suppose that for a particular student each homework assignment is marked with a pass with probability p and a fail with probability $q := 1-p$, independently of the other assignments. If we denote by $y_{n,k,p}$ the probability that the student qualifies for the final exam, determine $y_{n,k,p}$.*

In the following discussion, we consider some particular fixed p and, when the value of k is clear, we suppress the p and k indices in $y_{n,k,p}$ for ease of notation.

Similarly to the non-probabilistic case, we try to obtain a recurrence formula by examining when the first P occurs.

Proposition 3.3. *For $k \geq 1$ we will prove that $(y_{n,k,p})_{n\geq 1}$ satisfies the following recurrence relation:*

$$y_{n,k,p} = \sum_{i=1}^{k} p\, q^{i-1} y_{n-i,k,p} \quad \textit{for all } n > k$$

with initial conditions $y_{n,k,p} = 1$ for $1 \leq n \leq k-1$, $y_{k,k,p} = 1-q^k$.

Proof. We shall make use of the following theorem from probability theory:

Theorem 3.4 (The Law of Total Probability). *Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space and let $B_1, B_2, \ldots, B_m \in \mathscr{F}$ be a finite partition of Ω (in other words, a set of non-empty pairwise disjoint events whose union is the entire sample space) such that $\mathbb{P}(B_i) > 0$ for all i. Then, for any event A we have*

$$\mathbb{P}(A) = \sum_{i=1}^{m} \mathbb{P}(A \mid B_i)\,\mathbb{P}(B_i).$$

Let A be the event that the student qualifies for the final exam. Now, using the theorem shown above we obtain:

$$\begin{aligned}
y_n &= \mathbb{P}(A) \\
&= \sum_{i=1}^{k} \mathbb{P}(A \mid \text{first } P \text{ on homework } i) \times \mathbb{P}(\text{first } P \text{ on homework } i) \\
&+ \mathbb{P}(A \mid \text{first } P \text{ after homework } k) \times \mathbb{P}(\text{first } P \text{ after homework } k).
\end{aligned}$$

We notice that $\mathbb{P}(A \mid \text{first } P \text{ after homework } k) = 0$ because, by definition A and 'first P after homework k' cannot happen simultaneously. Also, we note that $\mathbb{P}(\text{first } P \text{ on homework } i) = q^{i-1} \cdot p$ by independence of the marks, since we need a fail on the first $i-1$ assignments, each with probability q, and a pass in the i^{th} assignment, with probability p. Our equation becomes:

$$\mathbb{P}(A) = \sum_{i=1}^{k} \mathbb{P}(A \mid \text{first } P \text{ on homework } i) \cdot q^{i-1} \cdot p.$$

We know that the remaining grades after a P behave the exact same way as a new sequence of grades so we naturally deduce that $\mathbb{P}(A \mid \text{first } P \text{ on homework } i) = y_{n-i}$. We get the desired result:

$$y_n = \sum_{i=1}^{k} y_{n-i}\, q^{i-1} p. \quad (2)$$

$\square$

To conclude this section, we note that if we choose $p = 0.5$ then we can recover the results of section 3.1. We can do this by noting

$$y_{n,k,0.5} = \frac{x_{n,k}}{2^n} \quad \text{for all } n,\, k \geq 1.$$

This formula holds since all the possible arrangements in $x_{n,k}$ have the same probability of happening in this case, namely $\frac{1}{2^n}$. In this sense, this section is a true generalization of the discussion in section 3.1.

3.5. Finding an Explicit Formula for $y_{n,2,p}$

In this section we will write y_n again instead of $y_{n,2,p}$ for ease of notation.

Subject to the following initial conditions:

$$\begin{cases} y_1 = 1 \\ y_2 = 1 - q^2 = p^2 + 2pq \end{cases}$$

we look to solve:

$$y_n = p y_{n-1} + p q y_{n-2} \tag{2}$$

We shall only consider the case $p > 0$ as the case $p = 0$ is trivial and gives $y_n = 0$ for all $n \geq 2$.

We will try a solution to (2) of the type $y_n = t^n$ for all $n \geq 1$ for some $t \in \mathbb{R}^*$. This satisfies (2) if and only if:

$$t^n = p t^{n-1} + p q t^{n-2} \quad \text{for all } n \geq 3.$$

We divide by $t^{n-2} \neq 0$ and we obtain $t^2 = pt + pq$. This quadratic has discriminant $\Delta = p^2 + 4pq > 0$. We obtain two solutions $\Phi := \frac{p+\sqrt{p^2+4pq}}{2}$ and $\Psi := \frac{p-\sqrt{p^2+4pq}}{2}$ that satisfy the recurrence relation and we will now try to combine them to satisfy the initial conditions.

Hence we try a solution to (2) of the form $y_n = A \cdot \Phi^n + B \cdot \Psi^n$.

We look for A and B such that the initial conditions are satisfied. We get:

$$\begin{cases} A\cdot\Phi + B\cdot\Psi = 1 \\ A\cdot\Phi^2 + B\cdot\Psi^2 = p^2+2pq \end{cases}$$

$$\Leftrightarrow \begin{cases} A = \frac{1-B\Psi}{\Phi} \\ \frac{1-B\Psi}{\Phi}\cdot\Phi^2 + B\Psi^2 = p^2+2pq. \end{cases}$$

The second equation simplifies to $B\cdot\Psi(\Psi-\Phi) = p^2+2pq-\Phi$. After plugging in the explicit formulas for Ψ and Φ we get

$$B\left(\sqrt{p^2+4pq}-p\right)\sqrt{p^2+4pq} = 2p^2+4pq-p-\sqrt{p^2+4pq}$$

which reduces to

$$B = \frac{2p^2+4pq-p-\sqrt{p^2+4pq}}{\left(\sqrt{p^2+4pq}-p\right)\sqrt{p^2+4pq}} = \frac{p^2+3pq-\sqrt{p^2+4pq}}{\left(\sqrt{p^2+4pq}-p\right)\sqrt{p^2+4pq}}.$$

Plug this into $A = (1-B\Psi)/\Phi$ to get

$$A = \frac{p^2+3pq+\sqrt{p^2+4pq}}{\left(\sqrt{p^2+4pq}+p\right)\sqrt{p^2+4pq}}.$$

In the end, we obtain:

$$\begin{aligned} y_n = {} & \frac{p^2+3pq+\sqrt{p^2+4pq}}{\left(\sqrt{p^2+4pq}+p\right)\sqrt{p^2+4pq}}\left(\frac{p+\sqrt{p^2+4pq}}{2}\right)^n \\ & + \frac{p^2+3pq-\sqrt{p^2+4pq}}{\left(\sqrt{p^2+4pq}-p\right)\sqrt{p^2+4pq}}\left(\frac{p-\sqrt{p^2+4pq}}{2}\right)^n \quad \text{for all } n\geq 1. \end{aligned}$$

3.6. Infinite Homework

We now consider the impossible but mathematically interesting case in which the number of homework assignments n is infinite. More precisely, we try to solve the following problem:

Problem (Infinite Homework)**.** *Let k be a positive natural number, $p \in [0,1]$. A university course consists of an infinite number of lectures labeled by the natural numbers, and the professor gives homework to his students at each lecture. Each homework will be marked by the professor by either P (pass) or F (fail). In order to qualify for the final exam, a student must have no k consecutive homework fails and no missing homework. Suppose that for a particular student each homework assignment is marked with a pass with probability p and a fail with probability $q := 1-p$, independently of the other assignments. If we denote by $y_{\infty,k,p}$ the probability that the student qualifies for the final exam, determine $y_{\infty,k,p}$.*

Proposition 3.5. *We have $y_{\infty,k,p} = 0$ if $p < 1$ and $y_{\infty,k,p} = 1$ if $p = 1$.*

Proof. As before, in what follows we write y_∞ for $y_{\infty,k,p}$ to ease notation.

Clearly, if $p = 1$ the student always receives a P so he always qualifies for the exam. Hence, $y_\infty = 1$.

Now suppose $p < 1$ and let A be the event that the student qualifies for the final exam. We use the same technique as in Proposition 3.3 to obtain that:

$$\begin{aligned} y_\infty &= \mathbb{P}(A) \\ &= \sum_{i=1}^{k} \mathbb{P}(A \mid \text{first } P \text{ on homework } i) \times \mathbb{P}(\text{first } P \text{ on homework } i) \end{aligned}$$

$$
\begin{aligned}
&+ \mathbb{P}(A \mid \text{first } P \text{ after homework } k) \times \mathbb{P}(\text{first } P \text{ after homework } k) \\
&= \sum_{i=1}^{k} \mathbb{P}(A \mid \text{first } P \text{ on homework } i) \times q^{i-1} p + 0 \times q^k \\
&= \sum_{i=1}^{k} y_\infty q^{i-1} p
\end{aligned}
$$

where we have used the fact that after we receive a pass, by independence, we find ourselves back where we started, with an infinite amount of homework ahead of us which we cannot fail k times consecutively. Hence, as $p = 1 - q$:

$$0 = \left(1 - p \sum_{i=1}^{k} q^{i-1}\right) y_\infty = \left(1 - p \frac{1 - q^k}{1 - q}\right) y_\infty = q^k y_\infty.$$

Thus, $y_\infty = 0$ since $q^k = (1-p)^k > 0$.

□

4. Conclusion

The original problem presented above, although simple in appearance, is full of interesting avenues to explore. In this article we have presented just a few of them but one can imagine a lot of other different generalizations both to the original problem and to some of the results presented above. In particular, we note that the results in sections 3.3 and 3.5 can be extended using the same method for $k > 2$ if one wishes to obtain explicit formulas.

This project has given us the opportunity to learn about recurrence relations, the Fibonacci numbers, probability and mathematical modelling as well as how to use LaTeX.

Editing Notes

(1) Indeed, this is the standard method for solving a linear recurrence equation like (1): any linear combination $(A\phi^n + B\psi^n)_{n\geq 0}$ of two solutions still satisfies the recurrence equation, and if moreover the first two terms are equal to those of $(F_n)_{n\geq 0}$, it will follow by induction that $F_n = A\phi^n + B\psi^n$ for all $n \geq 0$.

(2) It remains to verify that the initial values are those given in the statement of the proposition, $y_{n,k,p} = 1$ for $1 \leq n \leq k-1$ and $y_{k,k,p} = 1 - q^k$. The argument is the same as in section 3.1: for $1 \leq n \leq k-1$, we have $y_n = 1$ since it is impossible to receive k consecutive fails, and for $n = k$, $y_n = 1 - q^k$ since the student qualifies for the final exam unless he/she fails all k homeworks.